ANSWER KEY

M.Sc Mathematics

Ordinary Differential Equations

Course Code: 24PMASMT106 Batch: 2025-2071

Semester - I

Section-A

(1)

$$y''+16y=0$$

$$p(r)=r^2+16$$

$$r^2+16=0$$

$$r=\pm 4i$$

$$\phi(x)= c_1\cos 4x+c_2\sin 4x.$$

(2) Two functions ϕ_1, ϕ_2 defined on an interval I are said to be linearly dependent on I if there exists two constants c_1, c_2 not both zero such that $c_1\phi_1(x)+c_2\phi_2(x)=0$ for all x in I.

The functions ϕ_1, ϕ_2 are said to be linearly independent on I if there exists two constants c_1, c_2 such that $c_1\phi_1(x)+c_2\phi_2(x)=0$ for all x in I are the constants $c_1=0, c_2=0$.

(3)

$$\phi_1(x)=x^2 \qquad \phi_2(x)=5x^2$$

$$W(\phi_1,\phi_2)(x)=\begin{vmatrix}\phi_1(x) & \phi_2(x)\\ \phi_1'(x) & \phi_2'(x)\end{vmatrix}$$

$$=\begin{vmatrix}x^2 & 5x^2\\ 2x & 10x\end{vmatrix}=10x^3-10x^3=0$$

$\therefore$ the functions are not linearly independent.

4. $\phi_1(x) = e^x \quad \phi_2(x) = xe^x \quad \phi_3(x) = e^{-2x}$

$$W[\phi_1, \phi_2, \phi_3](x) = \begin{vmatrix} e^x & xe^x & e^{-2x} \\ e^x & xe^x + e^x & -2e^{-2x} \\ e^x & xe^x + 2e^x & 4e^{-2x} \end{vmatrix}$$

$$= e^x\left[4e^{-2x}(xe^x + e^x)\right] - xe^x\left[4e^{-x} + 2e^{-x}\right]$$
$$+ e^{-2x}\left[e^x(xe^x + 2e^x) - e^x(xe^x + e^x)\right]$$

$$= \cancel{4x} + 4 - \cancel{4x} - 2x + \cancel{x} + 2 - \cancel{x} - 1$$
$$= 5 - 2x \quad //$$

5. $y''' - 8y = 0$

$p(r) = r^3 - 8$

$(r-2)(r^2 + 2r + 4) = 0$

$r = 2, \ -1 \pm i\sqrt{3}$

$\phi(x) = C_1 e^{2x} + C_2 e^{(-1+i\sqrt{3})x} + C_3 e^{(-1-i\sqrt{3})x}$

6. Existence Theorem

Let $a_1, \ldots a_n$ be Continuous functions on an interval I Containing the point x_0. If $\alpha_1, \ldots \alpha_n$ are any n Constants, there exists a solution ϕ of $L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0$ on I satisfying $\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2 \ldots \phi^{n-1}(x_0) = \alpha_n$

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$$x^2y'' - 7xy' + 15y = 0$$

$$\phi_1(x) = x^3$$

$$\phi_1'(x) = 3x^2$$

$$\phi_1''(x) = 6x$$

$$x^2y'' - 7xy' + 15y$$

$$= x^2(6x) - 7x(3x^2) + 15x^3$$

$$= 6x^3 - 21x^3 + 15x^3$$

$$= 21x^3 - 21x^3 = 0$$

$\therefore$ ϕ_1 is the solution of an eqn.

8

The polynomial P_n of degree n of $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ satisfying $P_n(1) = 1$ is called the nth Legendre polynomial.

9

$$q(r) = r(r-1) + 2r - 6$$

$$= r^2 - r + 2r - 6$$

$$= r^2 + r - 6$$

$$r = 2, -3.$$

$$\phi_1(x) = |x|^2, \quad \phi_2(x) = |x|^{-3}.$$

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$$x^2y'' + (x+x^2)y' - y = 0$$

$x = 0$ is a singular point

$$x^2y'' + x(x+1)y' - y = 0$$

$1+x$, -1 are analytic at $x=0$

$\therefore$ $x=0$ is a regular singular point.

(11) $2xy\,dx + (x^2+3y^2)\,dy = 0$

$M = 2xy$, $N = x^2+3y^2$

$\frac{\partial M}{\partial y} = 2x$, $\frac{\partial N}{\partial x} = 2x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ eqn is exact.

(12) Let f be a function defined for (x,y) in a set S. We say that f satisfies a Lipschitz Condition on S if there exists a Constant $k>0$ such that $|f(x,y_1) - f(x,y_2)| \le k|y_1-y_2|$ $\forall$ (x,y_1) (x,y_2) in S. The Constant k is called a Lipschitz Constant.

Section-B

(13) let x_0 be a point in I. Since ϕ_1, ϕ_2 are linearly independent on I

$\Rightarrow W(\phi_1,\phi_2)(x_0) \neq 0$.

Let ϕ be any other Solution with $\phi(x_0)=\alpha$ $\phi'(x_0)=\beta$ & Consider the two equations

$c_1\phi_1(x_0) + c_2\phi_2(x_0) = \alpha$

$c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = \beta$

$\because W(\phi_1,\phi_2)(x_0) \neq 0$. There is a unique pair of Constants c_1, c_2 Satisfying these equations. If c_1, c_2 be these Constants then the function $\psi = c_1\phi_1 + c_2\phi_2$ is Such that

$\psi(x_0) = c_1\phi_1(x_0) + c_2\phi_2(x_0) = \alpha = \phi(x_0)$

$\psi'(x_0) = c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = \beta = \phi'(x_0)$

$L(\psi) = c_1L(\phi_1) + c_2L(\phi_2) = 0$ From the uniqueness theorem $\psi = \phi$

$\therefore \phi = c_1\phi_1 + c_2\phi_2$ //

(14) $y'' - 2y' - 3y = 0, \quad y(0) = 0, \quad y'(0) = 1$

The char polynomial

$r^2 - 2r - 3 = 0$

$r = -1, 3$

$y(x) = \phi(x) = c_1 e^{-x} + c_2 e^{3x}$

$y'(x) = -c_1 e^{-x} + 3c_2 e^{3x}$

$y(0) = 0 \implies c_1 + c_2 = 0$

$y'(0) = 1 \implies -c_1 + 3c_2 = 1$

$c_1 = -1/4, \quad c_2 = 1/4$

$\therefore \phi(x) = 1/4\, e^{3x} - 1/4\, e^{-x}$ //

(15) Ans: $\psi(x) = 3x/4\, e^{2x} - 4/3\, e^{-x}$.

(16) $y'' - 2/x^2\, y = 0$

$\phi_1(x) = x^2$

set $\phi_2 = u\phi_1$.

$\phi_2(x) = \frac{-1}{3x}$.

but any constant times a solution is a solution hence $\phi_2(x) = -1/x$.

(17) $P_m(x)$ and $P_n(x)$ satisfy Legendre's eqn

$(1-x^2) P_n'' - 2x P_n' + n(n+1) P_n = 0$ — (1)

$(1-x^2) P_m'' - 2x P_m' + m(m+1) P_m = 0$ — (2).

(2) $\times P_n -$ (1) $\times P_m \implies d/dx \left[(1-x^2)(P_n P_m' - P_n' P_m)\right] = (n-m)(n+m+1) P_m P_n$

$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$ as $m \neq n$

18 Refer book

19 $\phi_1(x) = 1, \quad \phi_2(x) = 1 + x^2/2, \quad \phi_3(x) = 1 + x^2/2 + x^4/2\cdot4$

Section-C

20 Refer book for proof

21 $y'' - y = \dfrac{2}{1+e^x}$.

$\phi_1(x) = e^{-x}, \quad \phi_2(x) = e^x$.

$\psi_p(x = -e^{-x}\log(1+e^x) + e^x\left[\log\left(\dfrac{1+e^x}{e^x}\right) - e^{-x}\right]$

$\psi(x) = C_1 e^{-x} + C_2 e^x + e^x \log\left(\dfrac{1+e^x}{e^x}\right) - 1 - e^{-x}\log(1+e^x)$ //.

22 Refer book: for proof

23 Refer book for proof

24 $\phi_1(x) = |x|^{-1} \sum_{k=0}^{\infty} \dfrac{(-1)^k x^k}{(\lfloor k)^2}$

$\phi_2(x) = -2|x|^{-1} \sum_{k=1}^{\infty} \dfrac{(-1)^k}{(\lfloor k)^2} \left(1 + 1/2 + \cdots + 1/k\right) x^k + (\log|x|)\,\phi_1(x)$

25 Solution $x^3 - x^2 y + y^2 = C$ //.