Scheme

Max. 75 marks
Time: 3 Hrs.

Course Name : Mechanics

Section-A

Answer any 10 questions $10 \times 1 = 10$ marks

1. The configuration of a system specified by the n-generalized coordinates $q_1, q_2, \ldots, q_n$ and assume that there are k- lin. indep. equations of constraints of the form $\varphi_j(q_1, \ldots q_n, t) = 0$ is known as holonomic constraint.

2. Any bilateral constraint such that virtual work of the corresponding forces is zero for any virtual displacement which is consistent with the constraint.

3. $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \quad (i = 1, 2, \ldots, n).$$

4. Suppose $L(q, \dot{q}, t)$ contains all n $\dot{q}$'s but some of the q's say $q_1, \ldots q_k$ are missing from the Lagrangian. Then there k- coordinates are called ignorable coordinates.

5. The problem of finding a curve $y(x)$ between origin O and the point (x_1, y_1) s.t. a particle starting from rest at O and sliding down the curve without friction under the influence of a uniform gravitational field, will reach the end of the curve in a min. time.

6. The actual path in configuration space followed by a holonomic dynamical system during the fixed interval t_0 to t_1 is such that the ~~dynamical system during the fixed~~ ~~... ... ...~~ integral $\int_{t_0}^{t_1} L\,dt$ is stationary w.r.t the path variation which to vanish at the end-points

7 $$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

8. If $S(q, \alpha, t)$ is any complete soln of the Hamilton-Jacobi eqn. $\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0$ and if the equations $-\beta_i = \frac{\partial S}{\partial \alpha_i}$, $p_i = \frac{\partial S}{\partial q_i}$ $(i = 1, 2 \dots n)$ where the β's are arb. constants are used to solve for $q_i(\alpha, \beta, t)$ and $p_i(\alpha, \beta, t)$ then these expressions provide the general soln. of the canonical equations associated with the Hamiltonian $H(q, p, t)$.

9. It is the orthogonal system which has KE and PE of the form:

$$T = \frac{1}{2}\left[\sum f_i(q_i)\right]\left[\sum \frac{\dot{q}_i^2}{R_i(q_i)}\right] = \frac{R_1 p_1^2 + \dots + R_n p_n^2}{2(f_1 + \dots + f_n)}$$

$$V = \frac{V_1(q_1) + \dots + V_n(q_n)}{f_1(q_1) + \dots + f_n(q_n)}, \quad \text{where } f_i, R_i,$$

and V_i are each funcs. of q_i and R_i is identical with M_i^{-1} and $\sum f_i(q_i) > 0$ and $R_i(q_i) > 0$.

10. A transformation from (q, p) to (Q, P) which preserves the canonical form of the equations of motion is known as a canonical transformation.

11. A $2n \times 2n$ matrix M which satisfies

$$M^T Z M = \mu Z \quad \text{or} \quad M Z M^T = \mu Z$$ is known

as a symplectic matrix.

12 class of homo. canonical transformations for which a full set of n indep. $Q_j(q, Q, t)$ exist and are non zero.

Section - B

$5 \times 5 = 25$ marks

Answer any FIVE questions

13. Stat. The necessary and sufficient condition for the static equilibrium of an initially motionless scleronomic system which is subject to workless constraints is that zero virtual work be done by the applied forces in moving thro' an arb. virtual displacement satisfying the constraints.

pf. Consider a scleronomic system of N particles. If the system is in static equilibrium then for each particle, $F_i + R_i = 0$. Thus, the virtual work done by all the forces in moving thro' an arbitrary virtual displacement consistent with the constraints is zero.

$$\sum_{i=1}^{N} (F_i + R_i) \cdot \delta r_i = \sum_{i=1}^{N} F_i \cdot \delta r_i + \sum_{i=1}^{N} R_i \cdot \delta r_i = 0$$

If we assume all the constraints are workless, and if δr_i are reversible virtual displacements consistent with the constraints, then $\sum_{i=1}^{N} R_i \cdot \delta r_i = 0$.

and we conclude

$$\delta W = \sum_{i=1}^{n} F_i \cdot \delta r_i = 0.$$

Assume the same system of particles is initially motionless but not in equilibrium. Then one or more of the particles must have a net force applied to it, and in accordance with Newton's law of motion it will start to move in the direction of that force. Since any motion must be compatible with the constraints we can always choose a virtual displacement in the direction of the actual motion at each point. and in this case the virtual ~~displacement~~ work is positive. $\sum_{i=1}^{N} F_i \cdot \delta r_i + \sum_{i=1}^{N} R_i \cdot \delta r_i > 0$

Since constraints are workless, $\delta W = \sum_{i=1}^{N} F_i \cdot \delta r_i > 0$.

14. The general expression for the acceleration of a particle whose spherical coordinates are (r, θ, φ) is

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 \sin^2\theta)\, e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2 \sin\theta\cos\theta)\, e_\theta + (r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\, e_\varphi \quad (1)$$

where e_r, e_θ and e_φ are unit vectors forming an orthogonal triad.

A virtual displacement consistent with the instantaneous constraint is $\delta r = r\,\delta\theta\, e_\theta + r\sin\theta\,\delta\varphi\, e_\varphi$ (2)

Further, the gravitational force is

$$\vec{F} = -mg\cos\theta\, e_r + mg\sin\theta\, e_\theta \quad -③$$

Using ① ② & ③ in $\sum_{i=1}^{N}(F_i - m_i \ddot{r}_i)\cdot \delta r_i = 0$

we obtain

$$mr[g\sin\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta)]\,\delta\theta$$
$$- mr\sin\theta\,[r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta]\,\delta\varphi = 0$$

Since $\delta\theta$ and $\delta\varphi$ are indept virtual displacements, their coeffts must each be zero. Dividing out the common non-zero factors and sub. for r and its derivatives from $r = a + b\cos\omega t$ we get the diff. eqns of motion as follows:

$$(a+b\cos\omega t)\ddot{\theta} - 2b\omega\dot{\theta}\sin\omega t - (a+b\cos\omega t)\dot{\varphi}^2\sin\theta\cos\theta = g\sin\theta$$

$$(a+b\cos\omega t)\ddot{\varphi}\sin\theta - 2b\omega\dot{\varphi}\sin\omega t\sin\theta + 2(a+b\cos\omega t)\dot{\theta}\dot{\varphi}\cos\theta = 0.$$

15. Using spherical coord

$$KE \quad T = \tfrac{1}{2}m(l^2\dot{\theta}^2 + l^2\dot{\varphi}^2\sin^2\theta)$$

and $\quad PE \quad V = mgl\cos\theta$,

where the support point O is at the reference level corresponding to zero PE.

The Lagrangian fun. is $L = T - V$

$$= \tfrac{1}{2}ml^2[\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta] - mgl\cos\theta$$

for $q_i = \theta$, $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = ml^2\ddot{\theta}$; $\quad \frac{\partial L}{\partial\theta} = ml^2\dot{\varphi}^2\sin\theta\cos\theta + mgl\sin\theta$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \Rightarrow$$

The eqn of motion $\quad ml^2\ddot{\theta} - ml^2\dot{\varphi}^2\sin\theta\cos\theta - mgl\sin\theta = 0 \quad -(i)$

only $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\varphi}}\right) = ml^2\ddot{\varphi}\sin^2\theta + 2ml^2\dot{\theta}\dot{\varphi}\sin\theta\cos\theta$

$\frac{\partial L}{\partial\varphi} = 0$ and the $\dot{\varphi}$ equation of motion is

$$ml^2\ddot{\varphi}\sin^2\theta + 2ml^2\dot{\theta}\dot{\varphi}\sin\theta\cos\theta = 0 \quad -(ii)$$

(i) & (ii) are the reqd Eqns

16 Motion of a particle of unit mass which is attracted by an inverse square gravitational force to a fixed point O using polar coordinates, the KE $T = \frac{1}{2}(\dot{r}^2 + \dot{\theta}^2 r^2)$

and PE $= -\frac{\mu}{r}$

where μ is a positive const known as the gravitational coeff.

The Lagrangian fun is $L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu}{r}$

The r eqn of motion is

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0$$

Since θ doesn't appear in the Lagrangian fun, it is an ignorable coordinate.

The θ eqn of motion is $\frac{d}{dt}(r^2\dot{\theta}) = 0$

$\Rightarrow r^2\dot{\theta} = \beta$, β - const = angular momentum of the particle about the attracting center 'O'.

17 $F = z + \lambda_1[x^2+y^2+z^2-4] + \lambda_2[xy-1]$

$\frac{\partial F}{\partial x} = 2\lambda_1 x + \lambda_2 y = 0$; $\frac{\partial F}{\partial y} = 2\lambda_1 y + x\lambda_2 = 0$

$\frac{\partial F}{\partial z} = 1 + 2z\lambda_1 = 0$

Solving these eqns for $x, y, z, \lambda_1, \lambda_2$ we get four stationary points $(1, 1, \sqrt{2})$, $(-1, -1, \sqrt{2})$, $(1, 1, -\sqrt{2})$, $(-1, -1, -\sqrt{2})$.

First two are constrained max points of f and remaining two are constrained min points.

The Lagrangian multipliers are

$$\lambda_1 = -\frac{1}{2z} = \mp\frac{1}{2\sqrt{2}}, \quad \lambda_2 = \frac{1}{z} = \pm\frac{1}{\sqrt{2}}.$$

18. The KE $T = \frac{1}{2}m\dot{x}^2$; PE $V = \frac{1}{2}kx^2$

The momentum is $p = \frac{\partial T}{\partial \dot{x}} = m\dot{x}$

and Hamiltonian fun, is equal to the T.E.

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Since we're considering a conservative system,

By modified Hamilton-Jacobi eqn.

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}kx^2 = \alpha ,$$

α - energy const.

$$\frac{\partial W}{\partial x} = \sqrt{2m\left(\alpha - \frac{1}{2}kx^2\right)}$$

$$W(x,\alpha) = m\omega \int_{x_0}^{x} \sqrt{a^2 - \xi^2}\, d\xi , \quad a = \sqrt{2\alpha/m\omega^2}$$

$$\omega = \sqrt{k/m}$$

x_0 is either a convenient absolute const or a simple zero of $f(\xi)$, where $\sqrt{f(\xi)}$ is the integrand.

Applying $t - \beta_n = \frac{\partial W}{\partial \alpha_n}$ to the Ch. fn. (*)

Diff wrt α,

$$t - \beta = \frac{1}{\omega}\int_{x_0}^{x} \frac{d\xi}{\sqrt{a^2-\xi^2}} = \frac{1}{\omega}\left[\cos^{-1}\left(\frac{x_0}{a}\right) - \cos^{-1}\left(\frac{x}{a}\right)\right]$$

$$\Rightarrow \quad x = \sqrt{\frac{2\alpha}{m\omega^2}}\cos[\omega(t-t_0) - \varphi]$$

where $\cos\varphi = \frac{x_0}{a} = \sqrt{\frac{m\omega^2}{2\alpha}}\, x_0$ and $\beta = t_0$

If α is written in terms of initial conditions, $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$,

$$\alpha = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2 = \frac{m\omega^2}{2}\left[x_0^2 + \frac{v_0^2}{\omega^2}\right]$$

Also $\sin\varphi = \frac{1}{a}\sqrt{a^2 - x_0^2} = \frac{v_0}{a\omega}$

and $x = x_0\cos\omega(t-t_0) + \frac{v_0}{\omega}\sin\omega(t-t_0)$

19. The identity transformation

$$P_i = \frac{\partial F_2}{\partial q} = p_i \ ; \quad Q_i = \frac{\partial F_2}{\partial P_i} = q_i$$

$$\sum_{i=1}^{n} p_i\, dq_i - H dt - \sum_{i=1}^{n} P_i\, dQ_i + K dt = d\varphi \quad (i = 1, 2, \dots, n)$$

is associated with a canonical transformation from (q,p) to (Q,P). But the symmetry of this eqn. (*) the inverse of (*) (and $p_i = \frac{\partial \varphi}{\partial q_i}$, $P_i = -\frac{\partial \varphi}{\partial Q_i}$, $K = H + \frac{\partial \varphi}{\partial t}$ $i = 1, \dots, n$)

a given canonical transformation is itself canonical and is generated by then the negative of $G(q, Q, t)$.

Also, the sum of two exact differentials expressed in terms of (q, P, t) is exact. Hence, two canonical transformation performed in seqn. are equivalent to a single canonical transformation. So, the canonical transformations form a group for a given value of n.

Section - C

Answer any four questions $4 \times 10 = 40$ marks

20. Let a single particle whose position is given by (x, y, z) and total force F acting on the particle has the components

$$F_x = -\frac{\partial V}{\partial x}; \quad F_y = -\frac{\partial V}{\partial y}; \quad F_z = -\frac{\partial V}{\partial z}$$

, where $V(x, y, z)$ is a single-valued func of position only (not on velocity or time)

Consider the work dW done by F as it moves thro' an infinite small displacement $d\bar{r}$

We've
$$dW = \bar{F} \cdot d\bar{r} = F_x dx + F_y dy + F_z dz$$
$$= -\left[\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz\right]$$
$$= -dV(x, y, z).$$

$$W = \int_A^B \bar{F} \cdot d\bar{r} = \int_B^A dV = V_A - V_B.$$

Since the PE is a func. of position only, the work done on the particle depends upon the initial & final positions but independent of the specific paths connecting these points.

Thus, if A & B coincides, $W = 0 = \oint_C \bar{F} \cdot d\bar{r}$ for any conservative force $\bar{F}$.

21. Consider m non-holo. constraint eqns of the form

$$\sum_{i=1}^{n} a_{ji} dq_i + a_{jt} dt = 0 \qquad j = 1, 2, \ldots, n$$

The δq's must meet the following conditions since δq's are no longer indepen. If we assume a virtual displacements consistent with the constraints, must satisfy

$$\sum_{i=1}^{n} a_{ji}\, \delta q_i = 0 \qquad (j=1,2,\dots,m) \quad \text{---} (1)$$

Assume each generalized applied force Q_i is obtained from a potential fn. as $Q_i' = -\frac{\partial V}{\partial q_i}$

The constraints are assumed to be workless, so the generalized constraint forces C_i must meet $\sum_{i=1}^{n} C_i\, \delta q_i = 0$ --- (2)

For any virtual displacement consistent with the constraints.

multiply (1) by λ_j (Lagrange multiplier) to get m eqns

$$\lambda_j \sum_{i=1}^{n} a_{ji}\, \delta q_i = 0 \qquad (j=1,2,\dots m) \quad \text{---} (3)$$

(2) − (3) ⇒

$$\sum_{i=1}^{n} \left(C_i - \sum_{j=1}^{n} \lambda_j a_{ji} \right) \delta q_i = 0.$$

If we choose $C_i = \sum_{j=1}^{n} \lambda_j a_{ji} \quad (i=1,2,\dots n)$

then the coeffts of $\delta q_i = 0$. Then the δq's can be chosen independently.

With these assumptions, by equating the generalized force C_i with Q_i' using (2) and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i' \quad (i=1,\dots n)$

We get $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{n} \lambda_j a_{ji} \qquad (i=1,2,\dots n)$

the standard form of Lagrange's eqn. for a non-holo. system.

22. The actual path in configuration space followed by a holo. dyn. system during the fixed interval t_0 to t_1 is st. the integral $I = \int_{t_0}^{t_1} L\,dt$ is stationary wrt path variations which vanish at the end-points.

Pf Consider a system of N particles whose configuration rel. to an inertial frame is given by the vectors $r_1, r_2, \dots r_N$.

Using Lagrangian form of d'Alembert's principle we get

$$\sum_{i=1}^{N} (F_i - m_i \ddot{r}_i) \cdot \delta r_i = 0$$

where F_i is the applied force acting on the i^{th} particle. We assume the virtual displacements δr_i are reversible & consistent with the instantaneous constraints which are considered to be workless

variation in K E is $\delta T = \delta\left(\frac{1}{2}\sum_{i=1}^{N} m_i \dot{r}_i^2\right) = \left(\sum_{i=1}^{N} m_i \dot{r}_i\right) \cdot \delta \dot{r}_i$

But $\frac{d}{dt}\left[\sum_{i=1}^{N} m_i \dot{r}_i \cdot \delta r_i\right] = \sum_{i=1}^{N} m_i \ddot{r}_i \cdot \delta r_i + \sum m_i \dot{r}_i \cdot \delta \dot{r}_i$

operation of variation and of diff wrt time are interchangeable.

i.e $\frac{d}{dt}(\delta r_i) = \delta \dot{r}_i$.

and get $\int_{t_0}^{t_1} (\delta T + \delta W) dt = 0$ — (1)

After applying generalized coor. form of K.E, T, the virtual work is $\delta W = \sum_{i=1}^{n} Q_i \delta q_i$.

Using in (1) and applying $\delta W = -\delta V$ to get

$$\int_{t_0}^{t_1} \delta(T-V)\, dt = 0 \Rightarrow \delta \int_{t_0}^{t_1} L\, dt = 0$$

Deriving the soln of Brachistochrone problem & get the soln as

$x = a[1-\cos\theta]$ and $y = a[\theta - \sin\theta]$

23. Start with $y(x) = y^*(x) + \delta y(x)$.

represent $\delta y = \alpha\eta(x)$, $\eta(x)$ – arb. fun,

to get $y(\alpha, x) = y^*(x) + \alpha\eta(x)$.

assume $\eta(x_0) = \eta(x_1) = 0$. to get

$$\frac{dI}{d\alpha} = \int_{x_0}^{x_1} \Big[\frac{\partial f}{\partial y} \underbrace{\frac{\partial y}{\partial \alpha}}_{=\eta(x)} + \frac{\partial f}{\partial y'} \underbrace{\frac{\partial y'}{\partial \alpha}}_{=\eta'(x)}\Big] dx = 0.$$

Applying integration by parts for this

to get $\int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \eta(x)\, dx = 0$

$\eta(x)$ is arbitrary except for the restriction upon continuity, we set

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0.$$

for any curve $y = y^*(x)$.

the Euler-Lagrange eqn.

24. Orthogonal system is a separable system iff the following Stackel conditions are met.

that a non-singular $n \times n$ matrix $[\Phi_{ij}(q_i)]$ and a column matrix $\{\psi_i(q_i)\}$ exist s.t.

$$c^T \phi = (1, 0, \ldots 0)$$

and $c^T \psi = V$, where $V(q_1, \ldots q_n)$ is the PE and c is a column matrix composed of the n c's

pf for necessity of these conditions, assume that the given orthogonal system is separable, and therefore a ch. fun. $W(q, \alpha)$ which consists of the sum of terms of the form $W_i(q_i, \alpha_1, \ldots \alpha_n)$ as in $W = \sum_{i=1}^{n} W_i(q_i)$

This ch. fun. is a complete integral of the modified Hamilton-Jacobi eqn. $\frac{1}{2}\sum_{i=1}^{n} c_i \left(\frac{\partial W_i}{\partial q_i}\right)^2 + V = \alpha_i$, where we choose α_i as the TE. Because of the assumed separability we know that $\left(\frac{\partial W_i}{\partial q_i}\right)^2$ is a fun. of $(q_i, \alpha_1, \ldots \alpha_n)$ and furthermore, we can choose the separation const. st α's appear linearly. Hence we see that the most general form involving the single coordi q_i is

$$\left(\frac{\partial w_i}{\partial q_i}\right)^2 = -2\psi_i(q_i) + 2\sum_{j=1}^{n} \Phi_{ij}(q_i)\,\alpha_j$$

we get $-c^T\psi + c^T \Phi \alpha + V = \alpha_1$

Comparing the terms containing α's we find $c^T \phi = (1, 0, \ldots 0)$

$\Rightarrow c^T \psi = V$

proof for sufficiency condition

25 $p\delta q - P\delta Q = (p + \cot p)\,\delta q - q\cot^2 p\,\delta p$

$$\Rightarrow \frac{\partial}{\partial p}(p + \cot p) = \frac{\partial}{\partial q}(-q\cot^2 q)$$

$\therefore$ The transformation is canonical.

Get $F_1(q, Q) = q\cos^{-1}\sqrt{1 - q^2 e^{2Q}} + \sqrt{e^{-2Q} - q^2}$

$$\frac{\partial F_1}{\partial q} = \cos^{-1}\sqrt{1 - q^2 e^{2Q}} = p\,;$$

$$\frac{\partial F_1}{\partial Q} = -\sqrt{e^{-2Q} - q^2} = -P$$

$F_2 = qp + QP + q\cot p$

Using (q, P) as variable $\quad qp = q\tan^{-1}\left(\frac{q}{P}\right)$

$$QP = -P\log\sqrt{q^2 + P^2}$$

$$q\cot p = P$$

$\Rightarrow F_2(q, P) = q\tan^{-1}\left(\frac{q}{P}\right) + P\left(1 - \log\sqrt{q^2 + P^2}\right)$

$$\frac{\partial F_2}{\partial q} = \tan^{-1}\left(\frac{q}{P}\right) = p\,; \quad \frac{\partial F_2}{\partial P} = -\log\sqrt{q^2 + P^2} = Q$$

Third Gen. form $F_3(p, Q) = F_1 - qp = e^{-Q}\cos p$

$$\frac{\partial F_3}{\partial p} = -e^{-Q}\sin p = -q\,; \quad \frac{\partial F_3}{\partial Q} = -e^{-Q}\cos p = -P$$

and $F_4(p, P) = F_2 - qp = QP + q\cot p$

$$= P + P\log\left(\frac{\cos p}{P}\right)$$

$$\frac{\partial F_4}{\partial p} = -P\tan p = -q\,; \quad \frac{\partial F_4}{\partial P} = \log\left(\frac{\cos p}{P}\right) = Q.$$