ANNA ADARSH COLLEGE FOR WOMEN

24PMSMT108 - TOPOLOGY

ANSWER KEY

1.

Definition. A *topology* on a point set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$; that is, if $\{U_\alpha\}_{\alpha\in A} \subset \mathcal{T}$ then $\cup_{\alpha\in A} U_\alpha \in \mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$ is in $\mathcal{T}$; that is, if $U_1, U_2, \ldots, U_n \in \mathcal{T}$ then $\cap_{i=1}^n U_i \in \mathcal{T}$.

2.

Definition. Let X be a set. A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X (called *basis elements*) such that

(1) For each $x \in X$, there is at least one basis element $B \in \mathcal{B}$ such that $x \in B$.

(2) If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$ then there is $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_2 \cap B_2$.

The topology $\mathcal{T}$ *generated by* $\mathcal{B}$ is defined as: A subset $U \subset X$ is in $\mathcal{T}$ if for each $x \in U$ there is $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. (Therefore each basis element is in $\mathcal{T}$.)

3.

Definition. Define the *product mapping* $\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta$ for $\beta \in J$ as the function assigning to each element of $\prod_{\alpha\in J} X_\alpha$ its βth coordinate, $\pi_\beta((x_\alpha)_{\alpha\in J}) = x_\beta$. Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta\}$$

and let $\mathcal{S}$ denote the union of these collections, $\mathcal{S} = \cup_{\beta\in J} \mathcal{S}_\beta$. The topology generated by the subbasis $\mathcal{S}$ is the *product topology* and this topology on $\prod_{\alpha\in J} X_\alpha$ is the *product space*.

4.

Definition. If d is a metric on X then the collection of all ε-balls $B_d(x, \varepsilon)$ for $x \in X$ and $\varepsilon > 0$ is a basis for a topology on X, called the *metric topology* induced by d.

5.

Definition. Let X be a topological space. A *separation* of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. Space X is *connected* if there is no separation of X.

6.

Definition. Given topological space X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called *components* (or "connected components") of X.

7.

Definition. A space X is *limit point compact* if every infinite subset of X has a limit point.

8.

Definition. A topological space X is *locally compact at point x* if there is some compact subspace X of X that contains a neighborhood of x. If X is locally compact at each of its points, set X is *locally compact*.

9.

A space for which every open covering contains a countable subcovering is called a Lindelof space.

10.

Definition. A topological space X is *completely regular* if one-point sets are closed in X and if for each $x_0 \in X$ and each closed $A \subset X$ not containing x_0, there is a continuous $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

11.

Let X be a topological space. Let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$. If X is metrizable and $x \in \overline{A}$ then there is a sequence $\{x_n\} \subset A$ such that $\{x_n\} \to x$.

12.

Let X be a normal space. Let A be a closed subspace of X.

(a) Any continuous function of A into the closed interval $[a,b] \subset \mathbb{R}$ may be extended to a continuous function on all of X into $[a,b]$.

(b) Any continuous function of A into $\mathbb{R}$ may be extended to a continuous function on all of X into $\mathbb{R}$.

SECTION – B

13.

Proof. Let $U \times V$ be a basis element for the product topology on $X \times Y$. Then $(U \times V) \cap (A \times B)$ is a basis element for the subspace topology on $A \times B$. Now $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$. Since $U \cap A$ and $V \cap B$ are open relative to A and B, respectively, then $(U \cap A) \times (V \cap B)$ is a basis element for the product topology on $A \times B$. So the basis for the subspace topology on $A \times B$ is a subset of the basis for the product topology on $A \times B$. Conversely, a basis element for the product topology on $A \times B$ is of the form $(U \cap A) \times (V \cap B)$ where U and v are open in X and Y, respectively, and from the equality above, this is a basis element for the subspace topology on $A \times B$. So the basis for the product topology on $A \times B$ is a subset of the basis for the subspace topology on $A \times B$. So the bases are the same and, as claimed, the topologies are the same. □

14.

Proof (continued). (1)⇒(4) Let $x \in X$ and let V be a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is open since f is continuous and $x \in U$. That is, $f(U) \subset V$, as claimed.

(4)⇒(1) Let V be an open set of Y. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and so by hypothesis (4) there is open U_x in X with $x \in U_x$ and $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. Then with such open U_x chosen for each $x \in f^{-1}(V)$ we have $f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x$ and hence $f^{-1}(V)$ is open. Therefore, by the definition of continuous function, f is continuous and (1) follows. □

15.

Let $f : X \to Y$ be a continuum map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point $x \in X$ such that $f(c) = r$.

Proof. Suppose f, X, and Y are as hypothesized. The sets $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, +\infty)$ are disjoint (since $(-\infty, r)$ and $(r, +\infty)$ are disjoint) and nonempty since $f(a)$ is in one of these sets and $f(b)$ is in the other. Each is open in $f(X)$ under the subspace topology. ASSUME there is no point $c \in X$ such that $f(c) = r$. Then $f(X) = A \cup B$ and A and B form a separation of $f(X)$. But since X is connected and f is continuous then $f(X)$ is connected by Theorem 23.5, a CONTRADICTION. So the assumption that there is no such $c \in X$ is false and hence $f(c) = r$ for some $c \in X$. □

16.

Proof. Let Y be a closed subspace of the compact set X. Let $\mathcal{A}$ be an arbitrary open cover of Y by sets open in X. Let $\mathcal{B} = \mathcal{A} \cup \{A \setminus Y\}$. Then $\mathcal{B}$ is a covering of X by open sets and since X is compact then some finite subcollection of $\mathcal{B}$ covers X. This finite subcollection with $X \setminus Y$ removed (if $X \setminus Y$ is in the subcollection) is then a finite subcollection of $\mathcal{A}$ which covers Y. So by Lemma 26.1, Y is compact. □

17.

Proof. (a) Let X be Hausdorff. Let Y be a subspace of X with $x, y \in Y$. If U and V are disjoint neighborhoods of x and y (respectively) in X, then $U \cap Y$ and $V \cap Y$ are disjoint open neighborhoods of x and y (respectively) in Y (under the subspace topology).

Let $\{X_\alpha\}$ be a family of Hausdorff spaces. Let $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ be distinct points in $\prod X_\alpha$. Because $\mathbf{x} \neq \mathbf{y}$, there is some β such that $x_\beta \neq y_\beta$. Since X_β is Hausdorff there are disjoint open sets U and V in X_β with $x_\beta \in U$ and $y_\beta \in V$.

18.

Proof. Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$. Let $B_d(x, \varepsilon)$ be a basis element for the metric topology $\mathcal{T}$. By Lemma 13.3 (the (1)⇒(2) part) there is a basis element $\mathcal{B}' \subset B_d(x, \varepsilon)$. By Lemma 20.B, there is $B_{d'}(x, \delta) \subset B' \subset B_d(x, \varepsilon)$ and the first claim holds.

Suppose the δ/ε condition holds. Given a basis element B for the metric topology for $\mathcal{T}$ containing x, by Lemma 20.B there is a basis element $B_d(x, \varepsilon) \subset B$. By the hypothesized δ/ε condition there is $B' = B_{d'}(x, \delta) \subset B_d(x, \varepsilon) \subset B$. By Lemma 13.3 (the (2)⇒(1) part), $\mathcal{T}'$ is finer than $\mathcal{T}$. □

19.

Proof. Let X be metrizable with metric d. Let A and B be disjoint closed sets in X. For each $a \in A$, choose $\varepsilon_a > 0$ so that $B(a, \varepsilon_a)$ does not intersect B (since B is closed, it contains its limit points by Corollary 17.7, so a is not a limit point of B and such $B(a, \varepsilon_a)$ exists). Similarly, for each $b \in B$ choose $\varepsilon_b > 0$ so that $B(b, \varepsilon_b)$ does not intersect A. Define

$$U = \cup_{a \in A} B(a, \varepsilon_a/2) \text{ and } V = \cup_{b \in B} B(b, \varepsilon_b/2).$$

Then U and V are open sets and $A \subset U$, $B \subset V$. ASSUME $z \in U \cap V$. Then $z \in B(a, \varepsilon_a/2)$ and $z \in B(d, \varepsilon_b/2)$ for some $a \in A$ and $b \in B$. By the Triangle Inequality,

$$d(a, b) \leq d(a, z) + d(z, b) < \varepsilon_a/2 + \varepsilon_b/2.$$

If $\varepsilon_a \leq \varepsilon_b$ then $d(a, b) < \varepsilon_b$ and then $a \in B(b, \varepsilon_b)$, a CONTRADICTION.

Proof (continued) . Similarly, if $\varepsilon_b \leq \varepsilon_a$ then $d(a,b) < \varepsilon_a$ and $b \in B(a, \varepsilon_a)$, a contradiction. So the assumption that such $z \in U \cap V$ exists is false and U and V are disjoint open sets with $A \subset U$ and $B \subset V$. Therefore, X is normal. □

SECTION C

20.

Proof. Since the compliments of $\varnothing$ and X are X and $\varnothing$, respectively, then by definition of closed, both $\varnothing$ and X are closed (since X and $\varnothing$ are open) and (1) follows.

Given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$, we have be DeMorgan's law (see Munkres' page 11), $X \setminus \cap_{\alpha \in J} A_\alpha = \cup_{\alpha \in J}(X \setminus A_\alpha)$. Since the sets $X \setminus A_\alpha$ are open by definition, the right side of this equation is a union of open sets and so is open. Therefore the left hand side is open an so, by definition, its compliment $\cap_{\alpha \in J} A_\alpha$ is closed, as claimed in (2).

Proof (continued). If A_i is closed for $i = 1, 2, \ldots, n$, then again by DeMorgan's Law, $X \setminus \cup_{i=1}^n A_i = \cap_{i=1}^n (X \setminus A_i)$. The set on the right side is a finite intersection of open sets and is therefore open. So the left hand side is open and its compliment, $\cup_{i=1}^n A_i$, is closed, as claimed in (3). □

21.

Proof. (a) Let $f(x) = y_0$ for every $x \in X$. Let V be open in Y. Then $f^{-1}(V) = X$ if $y_0 \in V$ and $f^{-1}(V) = \varnothing$ if $y_0 \notin V$. In either case, $f^{-1}(V)$ is open and so f is continuous. (b) If U is open in X, then $j^{-1}(U) = U \cap A$ which is open in A (by definition of the subspace topology). (c) If U is open in Z then $g^{-1}(U)$ is open in Y since g is continuous and $f^{-1}(g^{-1}(U))$ is open in X since f is continuous. Now $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U) = f^{-1}(g^{-1}(U))$ and so $g \circ f$ is continuous. □

Proof. (d) The function $f|_A$ equals the composition of the inclusion map $j : A \to Y$ (which is continuous by part (b)) and $f : X \to Y$ (which is continuous by hypothesis). So by part (c), $f|_A$ is continuous.
(e) Let $f : X \to Y$ be continuous and $f(X) \subset Z \subset Y$. Let B be open in Z. Then (by definition) $B = Z \cap U$ for some open U in Y. Then

$$\begin{aligned} g^{-1}(B) &= g^{-1}(Z \cap U) = g^{-1}(Z) \cap g^{-1}(U) \\ &= X \cap g^{-1}(U) \text{since } f(X) = g(X) \subset Z \\ &= g^{-1}(U) \\ &= f^{-1}(U) \text{ since } f(x) \in Y \text{ for some } x \in X \text{ implies } g(x) = f(x) \in \end{aligned}$$

Since f is continuous, $f^{-1}(U)$ is open in X and so $g^{-1}(U)$ is open in X. Therefore, g is continuous.
Now let $h : X \to Z \supset Y$ be as described. Then h is the composition of $f : X \times Y$ (which is continuous by hypothesis) and the inclusion map $j : Y \to Z$ (which is continuous by part (b)). So, by part (c), h is continuous.

Proof. (f) Suppose $X = \cup_{\alpha \in J} U_\alpha$ for open U_α in X where $f|_{U_\alpha}$ is continuous for each $\alpha \in J$. Let V be an open set in Y. Since $f^{-1}(V) \cap U_\alpha$ consists of $x \in X \cap U_\alpha = U_\alpha$ such that $f(x) \in V$ and $(f|_{U_\alpha})^{-1}(V)$ consists of $x \in U_\alpha$ such that $f(x) \in U_\alpha$, then $f^{-1}(V) \cap U_\alpha = (f|_{U_\alpha})^{-1}(V)$ for all $\alpha \in J$. Since $f|_{U_\alpha}$ is continuous by hypothesis, then this set is open in U_α and since U_α is open then (by Lemma 16.2) this set is open in X. Since $X = \cup_{\alpha \in J} U_\alpha$ then

$$f^{-1}(V) = f^{-1}(V) \cap X = f^{-1}(V) \cap (\cup_{\alpha \in J} U_\alpha) = \cup_{\alpha \in J}(f^{-1}(V) \cap U_\alpha)$$

is open in X since each set in the union is open. Therefore (by definition) f is continuous. □

22.

Proof. Recall that a subspace Y of L is *convex* if for every pair of points $a, b \in Y$ with $a < b$, then entire interval $[a, b] = \{x \in L \mid a \le x \le b\}$ lies in Y.

Let Y be convex. ASSUME that Y has a separation and that Y is the union of disjoint nonempty sets A and B, each of which is open in Y. Choose $a \in A$ and $b \in B$. WLOG, say $a < b$. Since Y is convex then $[a, b] \subset Y$. Hence $[a, b]$ is the union of the disjoint sets $A_0 = A \cap [a, b]$ and $B_0 = B \cap [a, b]$, each of which is open in $[a, b]$ in the subspace topology on $[a, b]$ (since A and B are open in Y) which is the same as the order topology (by Theorem 16.4). Since $a \in A_0$ and $b \in B_0$, then $A_0 \neq \varnothing \neq B_0$ and so A_0 and B_0 form a separation of $[a, b]$.

Proof (continued). Let $c = \sup A_0$. We now show in to cases that $c \notin A_0$ and $c \notin B_0$, which CONTRADICTS the fact that $[a, b] = A_0 \cup B_0$ (since $A_0 \subset [a, b]$ then b is an upper bound for A_0 and so $a \leq c \leq b$ and so $c \in [a, b] = A_0 \cup B_0$). From this contradiction, it follows that Y is connected.

Case 1. Suppose $c \in B_0$. Then $c \neq a$ (since $a \in A$ and $A \cap B = \varnothing$). So either $c = b$ or $a < c < b$. In either case, since B_0 is open in $[a, b]$ then there is some interval of the form $(d, c] \subset B_0$. If $c = b$ we have a contradiction since this implies that d is an upper bound of A_0, but $d < c$. If $c < b$ we note that $(c, d] \cap A_0 = \varnothing$ since c is an upper bound of A_0. Then (with d as above where $(d, c] \subset B_0$) we have that $(d, b] = (d, c] \cup (c, b]$ does not intersect A_0. Again, d is a smaller upper bound of A_0 than c, a CONTRADICTION. We conclude that $c \notin B_0$.

Proof (continued).
Case 2. Suppose $c \in A_0$. Then $c \neq b$ since $b \in B$. So either $c = a$ of $a < c < b$. Because A_0 is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A_0. By property (2) of the linear continuum L, there is $z \in L$ such that $c < z < e$. Then $z \in A_0$, CONTRADICTING the fact that c is an upper bound of A_0. We conclude that $c \notin A_0$.

We have shown that if Y is a convex subset of L then Y is connected. Notice that intervals and rays are convex sets and so are connected. □

23.

Proof. We follow Munkres' 4-step proof.
Step 1. Let $a < b$ and let $\mathcal{A}$ be a covering of a, b be sets open in $[a, b]$ in the subspace topology (which is the same as the order topology, by Theorem 16.4). Let $x \in [a, b]$, $x \neq b$. If x has an immediate successor in X, let y be this immediate successor. Then $[x, y] = \{x, y\}$ and $[x, y]$ can be covered by at most two elements of $\mathcal{A}$. If x has no immediate successor in X, choose an element $A \in \mathcal{A}$ containing x. Because $x \neq b$ and A is open, A contains an interval of the form $[x, c)$ for some $x \in [a, b]$ (since this is an element of the basis for the order topology; see part (2) of the definition of "order topology"). Choose $y \in (x, c)$. Then the interval $[a, y]$ is covered by the single element A of $\mathcal{A}$. In either case, for each $x \in [a, b)$ there is $y > x$ where $y \in [a, b]$ such that $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.

Proof (continued).

Step 2. Let $C = \{y \in [a, b] \mid y > a$ and $[a, y]$ can be covered by finitely many elements of $\mathcal{A}\}$. Since $a \in C$, $C \neq \emptyset$. Let c be the least upper bound of set C (this is where the least upper bound property is used). Then, by Step 1, $a < c \leq b$.

Step 3. Since $\mathcal{A}$ is a covering of $[a, b]$, then some $A \in \mathcal{A}$ contains c. Since A is open, it contains an interval of the form $(d, c]$ for some $d \in [a, b]$ (see part (3) of the definition of "order topology"). ASSUME $c \not\in C$. Then there must be $x \in C$ with $z \in (d, c)$, otherwise $d < c$ would be an upper bound on C. Since $z \in C$, the interval $[a, z]$ can be covered by finitely many (say n) elements of $\mathcal{A}$ (by the definition of C). Now $[z, c] \subset (d, c] \subset A \in \mathcal{A}$, hence $[a, c] = [a, z] \cup [z, c]$ can be covered by $n + 1$ elements of $\mathcal{A}$. But then $c \in C$, a CONTRADICTION. So the assumption that $c \not\in C$ is false, and in fact $c \in C$.

Proof (continued).

Step 4. ASSUME $c < b$ where $c = \text{lub}(C)$, as defined in Step 2. By Step 1 with $x = c$, there is $y \in [a, b]$ with $y > c$ such that $[c, y]$ can be covered by finitely many elements of $\mathcal{A}$. From Step 3, $c \in C$ and so $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$. So $y \in C$. But $y > c$, CONTRADICTING the fact that c is an upper bound of C. So the assumption that $c < b$ is false and so $c = b$ (notice $c \leq b$ by Step 2). By Step 3, $b = c \in C$ and so the interval $[a, b]$ can be covered by finitely many elements of $\mathcal{A}$. Since $\mathcal{A}$ is an arbitrary open covering of $[a, b]$, then $[a, b]$ is compact. □

24.

Let X be a normal space. Let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map $f : X \to [a, b]$ such that $f(x) = a$ for every $x \in A$, and $f(x) = b$ for every $x \in B$.

Proof. Without loss of generality, we take $[a, b] = [0, 1]$. We use normality to construct a nested family of open sets which is indexed by the rational numbers in $[0, 1]$. Continuous f is then defined using the sets. We follow Munkres' four steps.

Step 1. let $P = [0, 1] \cap \mathbb{Q}$. Since P is countable, there is mapping $N : \mathbb{Q} \to \mathbb{N}$ which is a bijection. Take $N(1) = 1$ and $N(2) = 0$. We now define indexed sets $U_{N(p)}$ for $p \in P$. First, define $U_{N(1)} = U_1 = X \setminus B$. Second, because A is a closed set contained in open set U_1, by the normality of X, Lemma 31.1(b) implies that there is open $U_{N(2)} = U_0$ such that $A \subset U_0$ and $\overline{U_0} \subset U_1$.

Proof (continued). In general, let $P_n = \{N(1), N(2, \ldots, N(n)\}$ and suppose for $p, q \in P_n$ with $p < q$, we have already defined open U_p, U_q with $\overline{U}_p \subset U_q$. We now inductively define $U_{N(n+1)}$. Let $N(n+1) = r$ and $P_{n+1} = P_n \cup \{N(n+1)\} = P \cup \{r\}$. Now $r \neq 0, 1$ and so r has an immediate predecessor in P_{n+1}, say p, and an immediate successor in P_{n+1}, say q (this follows from Theorem 10.1). Since $p, q \in P_n$ then we are supposing that U_p and U_q are already defined with $\overline{U}_p \subset U_q$. By the normality of X, there is open $U_r \subset X$ such that $\overline{U}_p \subset U_r$ and $\overline{U}_r \subset U_q$ by Lemma 31.1(b). In this way, we have $U_{N(n)}$ defined for all $n \in \mathbb{N}$; that is, for all $p \in P$ we have defined open U_p. We claim that $p < q$ implies $\overline{U}_p \subset U_q$ for all $p, q \in P$. Let $p, q \in P_{n+1}$ with $p < q$. If $p, q \in P_n$ then $\overline{U}_p \subset U_q$ by the induction hypothesis. If one of p and q is r and the other is $s \in P_n$, then either $s \leq p$ in which case $\overline{U}_s \subset \overline{U}_p \subset U_r$ or $s \geq q$ in which case $\overline{U}_r \subset U_q \subset U_s$. Therefore, by induction, for any $p, q \in P$ we have $p < q$ implies $\overline{U}_p \subset U_q$. The sets are as illustrated in Figure 33.1.

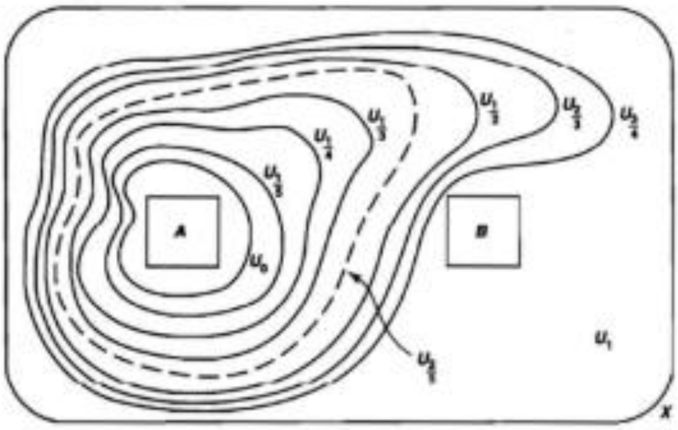

Figure 33.1

Step 2. In this step, we extend the definition to U_p from $p \in [0, 1] \cap \mathbb{Q}$ to all of $\mathbb{Q}$ by setting $U_p = \varnothing$ if $p < 0$ and $U_p = X$ if $p > 1$. We still have $p < q$ implying $\overline{U}_p \subset U_q$ for all $p, q \in \mathbb{Q}$.

Step 3. We now define f. For $x \in X$ define $\mathbb{Q}(x) = \{p \in \mathbb{Q} \mid x \in U_p\}$ where U_p is as defined above. Since $U_p = \varnothing$ for $p < 0$, for all $x \in X$ the set $\mathbb{Q}(x)$ contains no rationals less than 0. Since $U_p = X$ for $p > 1$, then all $x \in X$ are in U_p for $p > 1$. So $\mathbb{Q}(x)$ is bounded below and so has a greatest lower bound in $[0, 1]$. Define

$$f(x) = \inf \mathbb{Q}(x) = \inf\{p \in \mathbb{Q} \mid x \in U_p\}.$$

Step 4. If $x \in A$ then $x \in U_p$ for every rational $p \geq 0$ (since $A \subset U_p$ for all $p \geq 0$) and so $f(x) = 0$ for all $x \in A$, as desired. If $x \in B$, then $x \in U_p$ for no rational $p \leq 1$ but $x \in U_p = X$ for all rational $p > 1$. Hence $f(a) = 1$ for all $x \in B$, as desired.

Proof (continued). Now to show f is continuous. We first prove two things:

(1) If $x \in \overline{U}_r$, then $f(x) \leq r$.

(2) If $x \notin U_r$ then $f(x) \geq r$.

To prove (1), note that if $x \in \overline{U}_r$ then $x \in U_s$ for every $s > r$ (by the construction of the U_p in Step 1). Therefore $\mathbb{Q}(x)$ contains all rationals greater than r and so $f(x) = \inf \mathbb{Q}(x) \leq r$ for $x \in \overline{U}_r$. To prove (2), not that if $x \notin U_r$ then $x \notin U_s$ for any $s < r$ (since $\overline{U}_s \subset U_r$ for $s < r$). So $\mathbb{Q}(x)$ contains no rational numbers less than r, so that $f(x) = \inf \mathbb{Q}(x) \geq r$ for $x \notin U_r$.

Now given $x_0 \in X$ and open interval $(c, d) \subset \mathbb{R}$ containing $f(x_0)$, choose rational p and q such that $c < p < f(x_0) < q < d$. Consider $U = U_q \setminus \overline{Q}_p$. We have $f(x_0) < q$ so by the contrapositive of condition (2) we have that $x_0 \in U_q$. Since $f(x_0) > p$, the contrapositive of (1) implies that $x_0 \notin \overline{U}_p$. So $x_0 \in U = U_q \setminus \overline{U}_p$.

Proof (continued). Let $x \in U$. Then $x \in U_q \subset \overline{U}_q$ so that $f(x) \leq q$ by condition (1). Since $x_0 \notin \overline{U}_p$ then $x_0 \notin U_p$ and $f(x) \geq p$ by condition (2). Therefore, $f(x) \in [p.q] \subset (c, d)$. So $f(U) \subset (c, d)$ and $U = U_q \setminus \overline{U}_p - U_q \cap (X \setminus \overline{U}_p)$ is an open set containing x_0 such that $f(U) \subset (c, d)$. So f is continuous at arbitrary point $x_0 \in X$ and f is the desired function. □

25.

Proof. We prove the result for two connected spaces X and Y and then the general result follows by induction.

Choose $(a,b) \in X \times Y$. Then $X \times \{b\}$ and $\{x\} \times Y$ are connected (for each $x \in X$), as will be shown in the homework (Exercise 23.A). For each $x \in X$, define $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$. Then T_x is connected by Theorem 23.3 (since (x,b) is in each constituent space). Next, consider $\cup_{x \in X} T_x = X \times Y$ (see Figure 23.2 on page 151 for motivation). This union is connected by Theorem 23.3 since the point (a,b) is common to each T_x. That is, $X \times Y$ is connected.

The proof for any finite product of connected spaces follows by induction along with the fact (established in Exercise 23.B) that $X_1 \times X_2 \times \cdots \times X_n$ is homeomorphic with $(x_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$. □