Scheme of Valuation

Mathematics - I

Section-A (10 × 2 = 20 marks)

1. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$ — 2 marks.

2. $\frac{1+3x}{1!} + \frac{(1+3x)^2}{2!} + \frac{(1+3x)^2}{3!} + \cdots$

$= e^{1+3x} - 1 = e \cdot e^{3x} - 1$ — 2 marks

3. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

Put $x = 1$

$\log(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$

$\log 2 = 1 - 1/2 + 1/3 - 1/4 \cdots$ — 2 marks.

4. A matrix A is said to be symmetric if $A = A^T$ or $A = A'$ — 2 marks.

5. A matrix is said to be Unitary if $A = A^*$ where $A^* = (\bar{A})^T$ or $(\bar{A})'$ — 2 marks.

6. Every square matrix satisfies its own characteristic equation (Cayley Hamilton theorem) 2 marks.

7. $\Delta f(x) = f(x+h) - f(x)$

$= E f(x) - f(x)$

$= (E-1) f(x) \Rightarrow \Delta = E - 1$

$\Rightarrow 1 + \Delta = E$ — 2 marks.

8. Lagrange's interpolation formula:-

$$f(x) = y_0 \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} + y_1 \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} + \cdots + y_n \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$ — 2 marks

9. If $x = \cos\theta + i\sin\theta$

$1/x = \cos\theta - i\sin\theta$

$x - 1/x = 2i\sin\theta$

$(x-1/x)^n = (2i\sin\theta)^n = 2^n i^n \sin^n\theta$ — 2 marks.

10. $\cos^2 x + \sin^2 x = 1$

$\cos^2 ix + \sin^2 ix = 1$ (Put $x = ix$)

$\cosh^2 x + (i\sinh x)^2 = 1$

$\cosh^2 x + i^2\sinh^2 x = 1$

$\cosh^2 x - \sinh^2 x = 1$ —— 2 marks.

11. $y = (ax+b)^n$

$y_1 = n(ax+b)^{n-1} \cdot a$

$y_2 = n(n-1)(ax+b)^{n-2} \cdot a^2$

$\vdots$

$y_n = n(n-1) \cdots 1 (ax+b)^0 \cdot a^n = n!\,a^n$ — 2 marks

12. $x = r\cos\theta \quad y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$= r$ — 2 marks.

Section - B. $(5 \times 5 = 25$ marks$)$.

13. $$\sum_{n=0}^{\infty} \frac{5n+1}{(2n+1)!}$$

$$u_n = \frac{5n+1}{(2n+1)!} = \frac{-3/2 + 5/2(2n+1)}{(2n+1)!}$$

$$u_n = -\frac{3}{2}\left(\frac{1}{(2n+1)!}\right) + \frac{5}{2}\frac{1}{(2n)!}$$

$$u_0 = -\frac{3}{2}\frac{1}{1!} + \frac{5}{2}\frac{1}{0!}$$

$$u_1 = -\frac{3}{2}\frac{1}{3!} + \frac{5}{2}\frac{1}{2!}$$

$$u_2 = -\frac{3}{2}\frac{1}{5!} + \frac{5}{2}\frac{1}{4!}$$

$\cdot$ - - - - - - -

Adding u_0, u_1, u_2 - - -

$$S = -\frac{3}{2}\left(\frac{e-\frac{1}{e}}{2}\right) + \frac{5}{2}\left(\frac{e+\frac{1}{e}}{2}\right)$$

$$= -\frac{3}{4}e + \frac{3}{4e} + \frac{5}{4}e + \frac{5}{4e}$$

$$= \frac{2}{4}e + \frac{8}{4e} = \frac{e}{2} + \frac{2}{e}$$ — 5 marks.

14. A matrix 'A' is said to be orthogonal if $AA^T = A^T A = I$.

$$A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix} \quad A^T = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$AA^T = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^T A = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$ — 5 marks.

15. $$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow 3 - 4\lambda + \lambda^2 - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$4A = \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix}$$

$$5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \qquad A^2 - 4A - 5 = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

CHT is verified.

$$A^2 - 4A - 5I = 0 \Rightarrow A^{-1}(A^2 - 4A - 5I) = 0 \Rightarrow A - 4 = 5A^{-1}$$

$$\Rightarrow \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} = A^{-1}$$ 5 marks.

16)

$f(x) = x^3+3x-1 \qquad f'(x) = 3x^2+3$

$f(0) = -1 \quad -ve$

$f(1) = 3 \quad +ve.$

$\therefore$ Root of $f(x)=0$ lies between 0 & 1.

Let $x_0 = 0$.

Newton's formula $\Rightarrow x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$x_1 = 0 - \dfrac{(-1)}{3}$

$x_1 = 1/3$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 1/3 - \dfrac{1/27}{10/3}$

$= 0.3222$

$= 0.32$ correct to 2 decimals

— 5 marks

17) Backward difference table.

x	y	∇	∇^2	∇^3	∇^4
10	35.4	-3.2	0.1	-0.1	0.3
15	32.2	-3.1	0.0	0.2	
20	29.1	-3.1	0.2		
25	26.0	-2.9			
30	23.1				

$u = \dfrac{x - x_4}{h} = \dfrac{27-30}{5} = -0.6$

$y = y_4 + \dfrac{u}{1!}\nabla y_4 + \dfrac{u(u+1)}{2!}\nabla^2 y_4 + \dfrac{u(u+1)(u+2)}{3!}\nabla^3 y_4 + \dfrac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_4$

$= 23.1 + \left(\dfrac{-0.6}{1!}\right)(-2.9) + \dfrac{(-0.6)(-0.6+1)}{2!}(0.2) + \dfrac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(0.2)$

$+ \dfrac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4!}(0.3)$

$= 24.79472.$

— 5 marks

18) $\sin 6\theta = \mathrm{Im}(c+is)^6$

18) $\sin 6\theta = Im(c + is)^6$ $c = \cos\theta$, $s = \sin\theta$

$= Im(c^6 + 6c^5 is + 15c^4 i^2 s^2 + 20c^3 i^3 s^3 + 15c^2 i^4 s^4 + 6c i^5 s^5 + i^6 s^6)$

$= 6c^5 s - 20c^3 s^3 + 6cs^5$

$\frac{\sin 6\theta}{\sin\theta} = 6c^5 - 20c^3 s^2 + 6cs^4$

$= 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-\cos^2\theta)^2$

$= 6\cos^5\theta - 20\cos^3\theta + 20\cos^5\theta + 6\cos\theta + 6\cos^5\theta - 12\cos^3\theta$

$= 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$ — 5 marks.

19) $y = \sin^{-1} x$

$y_1 = \frac{1}{\sqrt{1-x^2}}$

$y_1^2 = \frac{1}{1-x^2} \Rightarrow y_1^2(1-x^2) = 1$

$\Rightarrow y_1^2(-2x) + (1-x^2)\, 2y_1 y_2 = 0$

$\div 2y_1 \Rightarrow -xy_1 + (1-x^2)y_2 = 0.$

$\Rightarrow (1-x^2)y_2 - xy_1 = 0.$

Diff n times & using Leibnitz's theorem, we get.

$(1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} - n^2 y_n = 0$ — 5 marks.

Section - C (3X10=30 marks)

20) $S = \frac{1}{3\cdot 6} + \frac{1\cdot 3}{3\cdot 6\cdot 9} + \frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9\cdot 12} + \cdots$

$= \frac{1}{1\cdot 2}\left(\frac{1}{3^2}\right) + \frac{1\cdot 3}{1\cdot 2\cdot 3}\left(\frac{1}{3^3}\right) + \frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 4}\left(\frac{1}{3^4}\right) + \cdots$

In Numerator there should be one more factor. The previous term is -1 (as they are in AP with common diff 2)

5.

Multiply by -1 and adding the first two factors on both sides we have

$$-S + 1 + \frac{-1}{1}\left(\frac{1}{3}\right) = 1 + \frac{-1}{1}\left(\frac{1}{3}\right) + \frac{-1\cdot 1\cdot}{1\cdot 2}\left(\frac{1}{3^2}\right) + \frac{-1\cdot 1\cdot 3}{1\cdot 2\cdot 3}\left(\frac{1}{3^3}\right) + \cdots$$

$$-S = 1 + \left(\frac{-1}{1}\right)\left(\frac{1}{3}\right) + \frac{-1\cdot 1}{1\cdot 2}\left(\frac{1}{3^2}\right) + \cdots - 1 - \left(\frac{-1}{1}\right)\left(\frac{1}{3}\right)$$

— 5 marks

$$p = -1 \quad q = 2 \quad \frac{x}{q} = \frac{1}{3} \quad x = \frac{2}{3}$$

$$-S = (1-x)^{-p/q} - \left[1 + \left(\frac{-1}{1}\right)\left(\frac{1}{3}\right)\right]$$

$$= \left(1 - \frac{2}{3}\right)^{+1/2} - \frac{2}{3}$$

$$= \left(\frac{1}{3}\right)^{+1/2} - \frac{2}{3}$$

$$\Rightarrow S = \frac{2}{3} - \frac{1}{\sqrt{3}}$$ — ~~10~~ 5 marks.

21)

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{bmatrix}$$

Characteristic eq is $\lambda^3 + \lambda^2 - 19\lambda + 26 = 0$. — 3 marks

$$A^3 = \begin{bmatrix} -16 & -21 & 45 \\ -43 & -16 & 67 \\ 67 & 45 & -104 \end{bmatrix} \quad A^2 = \begin{bmatrix} 9 & 2 & -7 \\ 5 & 9 & -10 \\ -10 & -7 & 21 \end{bmatrix}$$

$\Rightarrow A^3 + A^2 - 19A + 26 = 0$. [C. H. Theorem is verified].

— 5 marks.

$$A^{-1}(A^3 + A^2 - 19A + 26) = 0$$

$$A^2 + A - 19I + 26A^{-1} = 0.$$

$$A^{-1} = \frac{1}{26}\left\{-A^2 - A + 19I\right\} = \frac{1}{26}\begin{bmatrix} 9 & -1 & 5 \\ -3 & 9 & 7 \\ 7 & 5 & 1 \end{bmatrix}$$

— 2 marks.

22.

$x_0 = 0 \quad h = 1$

From Forward difference table

$y_0 = 176 \quad \Delta y_0 = 9 \quad \Delta^2 y_0 = 0 \quad \Delta^3 y_0 = 0$

$\Delta^4 y_0 = 0 \quad \Delta^5 y_0 = -1 \quad \Delta^6 y_0 = 5.$ — 5 marks

$$f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)(u-5)}{5!}\Delta^5 y_0$$
$$+ \frac{u(u-1)(u-2)(u-3)(u-4)(u-5)(u-6)}{6!}\Delta^6 y_0.$$

$$u = \frac{x - x_0}{h} = 0.2$$

$f(0.2) = 177.6723$ — 5 marks.

23)

$x = \cos\theta + i\sin\theta \qquad x + 1/x = 2\cos\theta \qquad x^n + 1/x^n = 2\cos n\theta$

$1/x = \cos\theta - i\sin\theta \qquad x - 1/x = i2\sin\theta \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$

— 2 marks.

$$(2\cos\theta)^5 (2i\sin\theta)^6 = (x + 1/x)^5 (x - 1/x)^6$$
$$= (x^2 - 1/x^2)^5 (x - 1/x)$$
$$= \left(x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right)\left(x - \frac{1}{x}\right)$$
$$= \left(x^{11} + \frac{1}{x^{11}}\right) - \left(x^9 + \frac{1}{x^9}\right) - 5\left(x^7 + \frac{1}{x^7}\right)$$
$$+ 5\left(x^5 + 1/x^5\right) + 10\left(x^3 + \frac{1}{x^3}\right) - 10\left(x + \frac{1}{x}\right).$$
$$= (2\cos 11\theta - (2\cos 9\theta) - 5(2\cos 7\theta)$$
$$+ 5(2\cos 5\theta) + 10(2\cos 3\theta) - 10(2\cos\theta)$$

Dividing by 2 & using $i^6 = -1$, we get the result

— 8 marks.

24) $f(x,y) = 2(x^2-y^2) - x^4 + y^4$

$f_x = 4x - 4x^3 = 4x(1-x^2)$

$f_y = -4y + 4y^3 = -4y(1-y^2)$

$f_{xx} = 4 - 12x^2$; $f_{yy} = -4 + 12y^2$

$f_{yx} = 0$. — 3 marks.

$f_x = 0 \Rightarrow x(x^2-1) = 0 \quad x = 0, 1, -1$

$f_y = 0 \Rightarrow y(y^2-1) = 0 \quad y = 0, 1, -1$

$\therefore$ The points are $(0,0)\ (0,1)\ (0,-1)\ (1,0)\ (1,1)\ (1,-1)$
$(-1,0)\ (-1,1)\ (-1,-1)$.

$\Delta = f_{xx} \cdot f_{yy} - (f_{yx})^2 = (4-12x^2)(-4+12y^2)$

$\Delta = -16(1-3x^2)(1-3y^2)$. — 2 marks

At $(0,0)\ (1,1)\ (1,-1)\ (-1,1)\ (-1,-1)$ Δ is $-$ve

$\therefore$ These are saddle points.

At $(0,1)\ (0,-1)$; Δ is $+$ve & f_{xx} is $+$ve

$\therefore f(x,y)$ is min and min value $= -1$

At $(1,0)$ & $(-1,0)$ Δ is $+$ve & f_{xx} is $-$ve.

$\therefore f(x,y)$ is Max & Max value $= 1$ — 5 marks.

—— X ——