24PMSMT104

Algebra – II

Section – A (10 × 1 = 10 marks)

1. K contains F i.e. $F \subset K$. $\longrightarrow$ (1) marks

2. which No is not algebraic No this called Transcendence No. $\longrightarrow$ (1) marks.

3. The dimension of K as a vector space over F $\longrightarrow$ (1) mark.

4. $a \in K$ is root of $p(x) \in F[x]$ of multiplicity m if $(x-a)^m \mid p(x) \Rightarrow (x-a)^{m+1} \nmid p(x)$ $\longrightarrow$ (1) marks.

5. The extension K of F is a simple extension of F if $K = F(\alpha)$ α in K $\longrightarrow$ (1) marks.

6. $f(x) \in F[x]$ E is an finite extension of F. $f(x)$ can be factored as a product of linear factor over E, not over any proper subfield of E. $\longrightarrow$ (1) mark.

7. G is group of automorphism of K, then the fixed field of G is the set all elements $a \in K$ such that $\sigma(a) = a$ $\forall$ $\sigma \in G$.

8. K is a normal extension of F if K is a finite extension of F such that F is the fixed field of $G(K, F)$. $\longrightarrow$ (1) mark.

9. If $a = 0$ is trivially true.
$p^m - 1 \Rightarrow a^{p^m - 1} = 1$ $a^{p^m} = a$ $\Big\} \longrightarrow$ (1) marks.

10. $x^{p^m} - x$ over J_p they are isomorphic $\longrightarrow$ (1) marks.

11. $x \in Q$ $N(x) = xx^*$ $\longrightarrow$ (1) marks.

12. $(\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)(\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2) = (\alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3)^2$
$+ (\alpha_0\beta_1 + \alpha_1\beta_0 + \alpha_2\beta_3 - \alpha_3\beta_2)^2 + (\alpha_0\beta_2 - \alpha_1\beta_3 + \alpha_2\beta_0 + \alpha_3\beta_1)^2$
$+ (\alpha_0\beta_3 + \alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_3\beta_0)^2$.
$\longrightarrow$ (1) marks. $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$.

Section - B (5x5 = 25 marks) (2)

13. $[L:F] = [L:K][K:F]$

$[L:K] = m$ $\quad [K:F] = n$ $\quad v_1, v_2 \ldots v_m$ be a basis of L over K

$w_1, w_2 \ldots w_n$ " " K over F. $\longrightarrow$ (1) marks.

$t = k_1 v_1 + \ldots + k_m v_m$

$= (f_{11} w_1 + \ldots + f_{1n} w_n) v_1 + \ldots + (f_{m1} w_1 + \ldots + f_{mn} w_n) v_m$ $\longrightarrow$ (2) marks

$f_{i1} w_1 + \ldots + f_{in} w_n = 0$ for $i = 1, 2 \ldots m$

(Pf) $\longrightarrow$ (2) marks.

14. $p(x)$ is of degree 1

$\alpha x + \beta = 0$

$p(a) = 0 \Rightarrow a = -\beta/\alpha$ $\longrightarrow$ (1) marks.

$(x-a)^m \mid p(x)$ $\quad m \leq n$ $\quad p(x) = (x-a)^m q(x)$ $\quad q(x) \in K[x]$ is of degree $n-m$.

$p(b) = (b-a)^m q(b)$

$q(b) = 0 \Rightarrow q(x)$ is degree $n-m < n$

$m + (n-m) = n$ in K $\longrightarrow$ (4) marks.

15. a, b in fixed field of G.

$\sigma(a) = a$ $\quad \sigma(b) = b$ $\quad \sigma(a \pm b) = \sigma(a) \pm \sigma(b)$

$= a \pm b$

$\sigma(ab) = \sigma(a) \cdot \sigma(b)$ $\quad$ verified

$= ab$

$\sigma(b^{-1}) = b^{-1}$

$\Rightarrow G$ is subfield of K $\longrightarrow$ (5) marks.

16. H leaves K_H $\quad H \subset G(K, K_H)$

$[K : K_H] \geq o(G(K, K_H))$

$o(G(K, K_H)) \geq o(H)$ $\quad [K : K_H] \geq o(G(K, K_H)) \geq o(H)$

$[K : K_H] = o(H)$

$h = (o(H))$ $\quad \alpha_1 = \sigma_1(a) + \sigma_2(a) + \ldots + \sigma_h(a) = \sum_{i=1}^{h} \sigma_i(a)$

$\alpha_2 = \sum_{i<j} \sigma_i(a) \sigma_j(a)$ $\longrightarrow$ (5) marks.

$\vdots$

$\alpha_h = \sigma_1(a) \sigma_2(a) \ldots \sigma_h(a)$

$p(x) = (x - \sigma_1(a))(x - \sigma_2(a)) \ldots (x - \sigma_h(a))$ $\quad o(H) \leq [K : K_H]$

$= x^h - \alpha_1 x^{h-1} + \alpha_2 x^{h-2} \ldots$

(3)

17. F has 2^n elements $x^{2n} = x$. F is a square.

$\alpha^{-1} = a^2$ for some $a \in F$

$1 + \alpha a^2 + \beta b^2 = 1 + \alpha\alpha^{-1} + 0 = 1 + 1 = 0$

F is an odd prime, F has p^n elements.

$W_\alpha = \{1 + \alpha x^2 \mid x \in F\}$

$1 + \alpha x^2 = 1 + \alpha y^2 \quad x = \pm y$

$1 + (p^n - 1)/2 = (p^n + 1)/2$

$W_\beta = \{-\beta x^2 \mid x \in F\} \quad p^n + 1/2$ elements.

→ (5) marks.

$c = 1 + \alpha a^2 \qquad c = -\beta b^2 \qquad 1 + \alpha a^2 + \beta b^2 = 0$

18. $xT_{\alpha^2} = xa^2 - a^2 x$

$2axa = 0 \qquad xT_a^4 = (xa^2 - a^2 x) - a^2(xa^2 - a^2 x)$

$= xa^4 - a^4 x \qquad xT_a^{2m}$

$$xT_a^p = xa^p - paxa^{p-1} + \frac{p(p-1)}{2} a^2 xa^{p-2} + \cdots - a^p x$$

$$p \Big| \frac{p(p-1)\cdots(p-i+1)}{i!}$$

→ (5) marks

$xT_a^p = xa^p - a^p x = xT_{ap} = xT_{ap^2}$.

19. $a \in D$ a is algebraic over C

$a^n + \alpha_1 a^{n-1} + \cdots + \alpha_{n-1} a + \alpha_n = 0 \qquad \alpha_1, \alpha_2 \ldots \alpha_n$ in C

$p(x) = x^n + \alpha_1 x^{n-1} + \cdots + \alpha_{n-1} x + \alpha_n$ in $C[x]$

$p(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$

→ (5) marks.

* product in a division ring is zero only if one of the terms of the product is zero.

$a - \lambda_k = 0 \qquad D = C$

Section — C

20 $f^{(i)}(x)$ to denote the i^{th} derivative of $f(x)$ with respect x.

(7)

$F(x) = f(x) + f^{(1)}(x) + f^{(2)}(x) + \cdots + f^{(r)}(x)$

$f^{(r+1)}(x) = 0$

$\frac{d}{dx}\left(e^{-x} F(x)\right) = -e^{-x} f(x)$

$$\frac{g(x_1) - g(x_2)}{x_1 - x_2} = g^{(1)}(x_1 + \theta(x_2 - x_1))$$

$F(1) - e F(0) = -e^{(1-\theta_1)} f(\theta_1) = \epsilon_1$

$\vdots$

$F(n) - e^n F(0) = -ne^{n(1-\theta_n)} f(n\theta_n) = \epsilon_n$

$c_n e^n + c_{n-1} e^{n-1} + \cdots + c_1 e + c_0 = 0$ → (10) marks.

$$f(x) = \frac{1}{(p-1)!} x^{p-1} (1-x)^p (2-x)^p \cdots (n-x)^p$$

$$\frac{(n!)^p}{(p-1)!} x^{p-1} + \frac{a_0 x^p}{(p-1)!} + \frac{a_1 x^{p+1}}{(p-1)!} + \cdots$$

$$|\epsilon_i| \leq e^n \frac{n^p (n!)^p}{(p-1)!}$$

As $p \to \infty$

$$\frac{e^n n^p (n!)^p}{(p-1)!} \to 0$$

p/o this is contradiction. e is transcendental.

21. $F[x]$ - ring of polynomials in x over F.

$V = (p(x))$ $F[x]$ generated by $p(x)$

V is a maximal ideal of $F[x]$ $E = F[x]/V$ is a field.

$\bar{F} = \{\alpha + V \mid \alpha \in F\}$

$F[x]/V = E \Rightarrow f(x)\psi = f(x) + V$

$1 + V, x + V, (x+V)^2 = x^2 + V, \ldots (x+V)^i$

$(x+V)^{n-1} = x^{n-1} + V$ (8) marks

$f(x) = \beta_0 + \beta_1 x + \cdots + \beta_k x^k$

$f(x)\psi = \beta_0\psi + (\beta_1\psi)(x\psi) + \cdots + (\beta_k\psi)(x\psi)^k$

$P(x)\psi = 0$ → 2 marks

$P(x)\psi = P(a)$ $a = x\psi$ E is a root of $P(x)$

22. $f(x)$ and $g(x)$ of degrees m and n, be the irreducible polynomials over F satisfied by a and b

root of $f(x)$ be $a = a_1, a_2 \ldots a_m$

$g(x)$ be $b = b_1, b_2 \ldots b_n$

$$\lambda = \frac{a_i - a}{b - b_j}$$

$\gamma \in F$ such that $a_i + \gamma b_j \neq a + \gamma b$ $\forall j$

$F(c) = F(a,b)$ $F(c) \subset F(a,b)$

$F(a,b) \subset F(c)$

$g(x)$ considered as a polynomial over $K = F(c)$ → 5 marks.

$h(x) = f(c - \gamma x)$ $h(b) = f(c - \gamma b) = 0$

$h(b_j) = f(c - \gamma b_j) \neq 0$

$(x-b)^2 \nmid g(x)$

$x - b$ is the greatest common divisor of $h(x)$ and $g(x)$ extension of K. → 5 marks.

(Proof) $F(a,b) = F(c)$

23. 1. $T = K_{G(K,T)}$
2. $H = G(K, K_H)$
3. $[K:T] = o[G(K,T)]$, $[T:F]$ = index of $G(K,T)$ in $G(K,F)$
4. T is a normal extension of F iff $G(K,T)$ is a normal subgroup of $G(K,F)$
5. when T is a normal extension of F, then $G(T,F)$ is isomorphic to $G(K,F) / G(K,T)$.

$G(K,F)$ – Galois group → 2 marks.

$f(x)$ be a polynomial in $F[x]$

$K_H = \{x \in H \mid \sigma(x) = x$ for every $\sigma \in H\}$

If for all → 8 marks.

24. A finite division ring (b) is necessarily a commutative field $\longrightarrow$ 2 marks.

K - finite division ring

$Z = \{z \in K \mid zx = xz \ \forall \ x \in K\}$

The No. of conjugate of a in K $(q^n - 1)/(q^{n(a)} - 1)$.

$$q^n - 1 = q - 1 + \sum_{\substack{n(a) \mid n \\ n(a) \neq n}} \frac{q^n - 1}{q^{n(a)} - 1}$$

$\phi_n(x) = \prod (x - \theta)$

$\phi_1(x) = x - 1$

$\phi_2(x) = x + 1$

$\phi_3(x) = x^2 + x + 1$

$\vdots$

$\phi_n(x) = \dfrac{x^n - 1}{g(x)}$

$f(x) = \prod_{\substack{k \mid n \\ k \neq d}} \phi_k(x)$

$\phi_n(x) \mid \dfrac{x^n - 1}{x^d - 1}$ $\longrightarrow$ 8 marks.

$\phi_n(q) \mid \dfrac{q^n - 1}{q^{n(a)} - 1}$

$|\phi_n(q)| = \prod |q - \theta| > q - 1$

25. Every positive integer can be expressed as the sum of square of four integers. $\longrightarrow$ 2 marks.

$n = x_0^2 + x_1^2 + x_2^2 + x_3^2$ $\quad x_0, x_1, x_2, x_3$ - Four integer.

$2 = 1^2 + 1^2 + 0^2 + 0^2$

$W_p = \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in J_p\}$

$V = \{x_0\delta + x_1 i + x_2 j + x_3 k \mid p$ divides all of $x_0, x_1, x_2, x_3\}$

$u \in H$, $\quad u = m_0\delta + m_1 i + m_2 j + m_3 k$

$2u = 2m_0\delta + 2m_1 i + 2m_2 j + 2m_3 k = (m_0 + m_0 i + m_0 j + m_0 k)$
$+ 2m_1 i + 2m_2 j + 2m_3 k = m_0 + (2m_1 + m_0) i + (2m_2 + m_0) j$
$+ (2m_3 + m_0) k$

8 marks.

$Ap = m_0^2 + (2m_1 + m_0)^2 + (2m_2 + m_0)^2 + (2m_3 + m_0)^2$

$y_0 = \dfrac{x_0 + x_1}{2} \quad y_1 = \dfrac{x_0 - x_1}{2} \quad y_2 = \dfrac{x_2 + x_3}{2} \quad y_3 = \dfrac{x_2 - x_3}{2}$

$\boxed{y_0^2 + y_1^2 + y_2^2 + y_3^2 = a}$