ANNA ADARSH COLLEGE FOR WOMEN (AUTONOMOUS)

End Semester Examination, Apr/May- 2026

SCHEME OF VALUATION

M.Sc., Mathematics	2025-2026	Semester: II
Real Analysis II	24PMSMT105	

SECTION- A : Qn. No. 1 To 12 : Answer ANY 10 : ONE MARK EACH

1.	A class of subsets of an arbitrary space X is said to be a σ-algebra if X belongs to the class and the class is closed under the formation of countable unions and of complements.	Measure Theory and Integration by G. de. Barra Page No. 30 Def: 3
2.	The set E is Lebesgue measurable if each set A we have $m^*(A) = m^*(A \cap E) + m^*(A \cap cE)$	P. No. 30 Defn: 2
3.	Let $\{f_n\}$ be a sequence of measurable function such that $\lvert f_n \rvert \le g$ where g is integrable and let $\lim f_n = f$ a.e. Then f is integrable and $\lim \int f_n dx = \int f dx$.	Page No. 63
4.	Since x^{-1} is a continuous function for $x>0$, and is measurable, it is positive. So the integral is defined. Also $\int_1^\infty x^{-1} dx > \int_1^n x^{-1} dx$. But $x^{-1} > k^{-1}$ on $[k-1,k)$, $\int_1^n x^{-1} dx > \sum_{k=2}^{n} \int_1^n k^{-1} \chi[k-1,k)\, dx > \sum_{k=2}^{n} k^{-1} \to \infty$ as $n \to \infty$	Page No. 59 Example: 5
5.	If $f(x)$ is any real function, $f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$ are said to be the positive and negative parts of f respectively	Page No. 61 Def: 4

6.	Let $S=\{\varphi_0, \varphi_1, \dots\}$ be a collection of functions in $L^2(I)$. If $(\varphi_n, \varphi_m)=0$ whenever $m \neq n$, the collection S is said to be an orthogonal system on I. If in addition each φ_n has norm 1, then S is said to be orthonormal on I	Mathematical Analysis by Tom M. APOSTOL Page No. 30 Def: 11.1
7.	Let f be real-valued and continuous on a compact interval $[a,b]$. Then for every $\varepsilon>0$, there is a polynomial p such that $\lvert f(x)-p(x)\rvert < \varepsilon$ for every x in $[a,b]$	Page No: 322 Theorem: 11.17
8.	If the limit $s(x)$ exists and if the Lebesgue integral $\int_0^{\delta} \frac{g(t)-s(x)}{t}\,dt$ exists for some $\delta < \pi$, then the Fourier Series generated by f converges to $s(x)$.	P. No. 319 Th: 11.13
10.	$h'(a) = f'(b) \circ g'(a)$ where $h = f \circ g$ and $b = g(a)$	Page No. 353 12.10
9.	The directional derivative of f at c in the direction u, denoted by $f'(c,u)$ is defined by $f'(c;u) = \lim_{h\to 0} \frac{f(c+hu)-f(c)}{h}$ whenever the limit on the right exists	Page No. 34 Def: 12.1
11.	If f is diff. at a and if $\nabla f(a)=0$, the point a is called a stationary point of f.	Page No. 377 Def: 13.9
12.	We have $f'(z) = D_1u + iD_1v$, so $\lvert f'(z)\rvert^2 = (D_1u)^2 + (D_2v)^2$. Also $J_f(z) = \det \begin{bmatrix} D_1u & D_2u \\ D_1v & D_2v \end{bmatrix} = (D_1u)^2 + (D_1v)^2$ by C-R equations.	P. No. 368 Th. 13.1

SECTION- B : Qn. No. 13 To 19 : Answer ANY 5 : FIVE MARKS EACH

13. It is enough to prove $m^*(A) \geqslant m^*(A \cap (-\infty, a)) + m^*(A \cap [a,\infty))$ for any set A.

Let $A_1 = A \cap (-\infty, a)$ and $A_2 = A \cap [a,\infty)$

For any $\epsilon > 0$, $\exists$ intervals I_n such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and

$$m^*(A) \geqslant \sum_{n=1}^{\infty} \ell(I_n) - \epsilon$$

$$\therefore m^*(A_1) + m^*(A_2) \leq \sum_{n=1}^{\infty} \ell(I_n') + \sum_{n=1}^{\infty} \ell(I_n'') \leq \sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \epsilon$$

Measure Theory and Integration by G. de Barra P. No. 32 Th: 6

14. Let the set P_n are measurable for each n and so $P = \bigcap_{n=1}^{\infty} P_n$ is measurable. Also $P^* = [0,1] - P$

$$= \bigcup_{n=1}^{\infty} \bigcup_{r=1}^{2^{n-1}} I_{n,r},$$

a union of disjoint sets. So $m(P^*) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$

P. No: 34 Ex: 8

15. Let $f = \liminf f_n$. For each measurable simple fun. φ with $\varphi \leq f$, we have $\int \varphi \, dx \leq \liminf \int f_n \, dx$

CASE 1: $\int \varphi \, dx = \infty$ To prove: $\liminf \int f_n \, dx = \infty$

CASE 2: $\int \varphi \, dx < \infty$. To prove: $\liminf \int g_n \, dx \geqslant \int \varphi \, dx$

Since $f_n \geqslant g_n$, we get $\int \varphi \, dx \leq \liminf \int f_n \, dx$

P. No: 57 Th: 3

16 Let us assume $f \in L(I)$, Then for each real β, we have,

$$\lim_{\alpha \to +\infty} \int_I f(t) \sin(\alpha t + \beta)\, dt = 0$$ Mathematical Analysis by T.M.Apostol

Proof: If f is the characteristic function of a compact interval $[a,b]$, the result is obvious since we have P.No. 313

$$\left| \int_a^b \sin(\alpha t + \beta)\, dt \right| = \left| \frac{\cos(\alpha t + \beta) - \cos(b\alpha + \beta)}{\alpha} \right| \leq \frac{2}{\alpha} \quad \text{if } \alpha > 0.$$ Th: 11.6

The result is valid if f is a step fun. Now we have to prove the result for every Lebesgue-integrable fun f.

(i) $\left| \int_I f(t) \sin(\alpha t + \beta) dt \right| \leq |\quad| + |\quad| \leq \int_I |f(t) - S(t)|\, dt + \varepsilon/2$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

17. If $v \to 0$ in the Taylor formula, the error term $\|v\| E_c(v) \to 0$, the linear term $T_c(v)$ also tends to 0 P.No.347 because if $v = v_1 u_1 + \cdots + v_n u_n$ where $u_1, u_2, \ldots u_n$ are the unit coordinate vectors, then by linearity we have Th. 12.4

$$T_c(u) = v_1 T_c(u_1) + \cdots + v_n T_c(u_n)$$ and each term on the RHS tends to 0 as $v \to 0$.

Let the Total derivative T_c can also be written as $f'(c)$

$$\therefore\ f(c+v) = f(c) + f'(c)(v) + \|v\| E_c(v)$$

where $E_c(v) \to 0$ as $v \to 0$

18. For every vector a in R^m, there is a point z in $L(x,y)$ such that $a \cdot \{f(y) - f(x)\} = a \cdot \{f'(z)(y-x)\}$ P.No.355 Th: 12.9

Proof: Let us consider $u = y - x$. Since S is open and $L(x,y) \subseteq S$, there is a $\delta > 0$ such that $x + tu \in S$ for all real t in the interval $(-\delta, 1+\delta)$

$$F(t) = a \cdot f(x+tu)$$

$$F'(t) = a \cdot f'(x+tu; u) = a \cdot \{f'(x+tu)(u)\}$$

By usual MVT, we have $F(1) - F(0) = F'(\theta)$ where $0<\theta<1$

Now $F'(\theta) = a \cdot \{f'(x+\theta u)(u)\} = a \cdot \{f'(z)(y-x)\}$

where $z = x+\theta u \in L(x,y)$

19. Let us write (x_1, x_2, x_3) instead of (x, y, z) and introduce the quadratic form P.No. 38?

Exaple.

$$q(x) = \sum_{j=1}^{3}\sum_{i=1}^{3} a_{ij}\, x_i x_j \quad \text{where} \quad x = (x_1, x_2, x_3)$$

and the $a_{ij} = a_{ji}$ are chosen so that the eq. of the surface become $q(x) = 1$. Finally, the semi axes of the quadric surface are $t_1^{-1/2}, t_2^{-1/2}, t_3^{-1/2}$.

SECTION- C : Qn. No. 20 To 25 : Answer ANY 4 : TEN MARKS EACH

G. De. Barra

P.No. 39

Th: 13

20. For each α, $[x : f(x)+c > \alpha] = [x : f(x) > \alpha - c]$, a measurable set. So $f+c$ is measurable.

If $c = 0$, cf is measurable.

If $c > 0$, $[x : cf(x) > \alpha] = [x : f(x) > c^{-1}\alpha]$ a measurable set.

If $c < 0$, $[x : cf(x) < \alpha]$ a measurable set

For fg, $\frac{1}{4}[(f+g)^2 - (f-g)^2]$ so it is sufficient to show that f^2 is measurable. If $\alpha < 0$, $[x : f^2(x) > \alpha] = R$ is measurable. If $\alpha \geq 0$, $[x : f^2(x) > \alpha]$ is a measurable set

In the same way we can prove $f+g$ and $f-g$ are measurable.

21. Let $\{f_n, n=1,2,\ldots\}$ be a sequence of non-negative measurable funs such that $\{f_n(x)\}$ is a monotone increasing for each x. Let $f = \lim f_n$. Then $\int f dx = \lim \int f_n dx$. P.No: 5 Th: 4

Proof: By Fatou's Lemma, we have

$$\int f dx = \int \lim\inf f_n \le \lim\inf \int f_n \, dx \rightarrow (1)$$

But $f \geqslant f_n$ by hypothesis. Hence by a theorem $\int f dx \geqslant \int f_n dx$ and hence $\int f dx \geqslant \lim\sup \int f_n dx \rightarrow (2)$

Hence from (1) and (2) we get $\int f dx = \lim \int f_n dx$.

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22. Define a function s by $s(x) = \lim\limits_{t\to 0^+} \dfrac{f(x+t)+f(x-t)}{2}$ P.No: 32 whenever the limit exists. Then for each x for which Th: 11.1 $s(x)$ is defined, the Fourier series generated by f is cesaro summable and has $(c,1)$ sum $s(x)$. We have

$$\lim_{n\to\infty} \sigma_n(x) = s(x)$$

Proof: Let $g_x(t) = [f(x+t)+f(x-t)]/2 - s(x)$, whenever $s(x)$ is defined. Then $g_x(t) \to 0$ as $t \to 0+$. Therefore, given $\varepsilon > 0$, there is a +ve $\delta < \pi$ such that $|g_x(t)| < \varepsilon/2$ wherever $0<t<\delta$. on $[0,\delta]$ we have

$$\left| \frac{1}{n\pi} \int_0^\delta g_x(t) \frac{\sin^2 \frac{1}{2}nt}{\sin^2 \frac{1}{2}t} dt \right| \le \frac{\varepsilon}{2n\pi} \int_0^\pi \frac{\sin^2 \frac{1}{2}nt}{\sin^2 \frac{1}{2}t} dt = \varepsilon/2$$

$$|\sigma_n(x) - s(x)| = \left| \frac{1}{n\pi} \int_0^\pi g_x(t) \frac{\sin^2 \frac{1}{2}nt}{\sin^2 \frac{1}{2}t} dt \right| < \varepsilon$$

In otherwords, $\sigma_n(x) \rightarrow s(x)$ as $n \rightarrow \infty$

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23. Assume that f and all its partial derivative of order $< m$ are differentiable at each pt of an open set S in $\mathbb{R}^n$. If a and b are two points of S such that $L(a,b) \subseteq S$, then there is a point z on the line segment $L(a,b)$ such that

$$f(b) - f(a) = \sum_{k=1}^{m-1} \frac{1}{k!} f^{(k)}(a; b-a) + \frac{1}{m!} f^{(m)}(z; b-a)$$ P.No: 361 Th: 12.14

Proof: Define g on $(-\delta, 1+\delta)$ by $g(t) = f[a + t(b-a)]$

Then $f(b) - f(a) = g(1) - g(0)$. We prove this theorem by applying one-dim Taylor formula to g.

$$g(1) - g(0) = \sum_{k=1}^{m-1} \frac{1}{k!} g^{(k)}(0) + \frac{1}{m!} g^{(m)}(\theta).$$

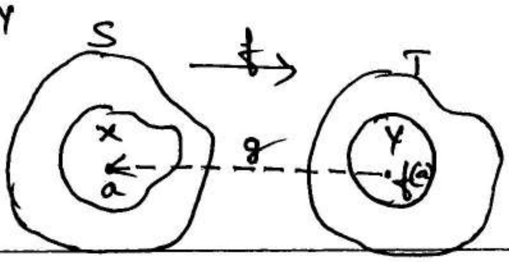
24. Assume $f = (f_1, \dots, f_n) \in C'$ on an open set S in $\mathbb{R}^n$ and let $T = f(S)$. If the Jacobian determinant $J_f(a) \neq 0$ for some point a in S, then there are two open sets $X \subseteq S$ and $Y \subseteq T$ and a uniquely determined fun g such that P.No: 372 Th: 13.3

(a) $a \in X$ and $f(a) \in Y$ (b) $Y = f(X$ (c) f is 1-1 on X

(d) g is defined on Y, $g(Y) = X$ and $g[f(x)] = x$ for every x in X

(e) $g \in C'$ on Y

Proof:



25. Let

$$Q(t) = \frac{1}{2} f''(a; t) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j} f(a) t_i t_j$$ Page No. 378 Th: 13.10

(a) If $Q(t) > 0$ for all $t \neq 0$, f has a relative min at a.

(b) If $Q(t) < 0$ for all $t \neq 0$, f has a relative max at a

(c) If $Q(t)$ takes both positive and negative values, then f has a saddle point at a.

Proof:

The function Q is continuous at each point t in R^n.

Let $S = \{t : \|t\| = 1\}$ denote the boundary of the n-ball $B(0;1)$

If $Q(t) > 0$, then $Q(t)$ is positive on S. Since S is compact, Q has a minimum on S and $m > 0$.

$$f(a+t) - f(a) = Q(t) + \|t\|^2 E(t) \geq m\|t\|^2 + \|t\|^2 E(t)$$

$$f(a+t) - f(a) > m\|t\|^2 - \tfrac{1}{2} m\|t\|^2 = \tfrac{1}{2} m\|t\|^2 > 0$$

$\therefore$ f has a relative minimum at a which proves (a).

To prove the second result we use the similar argument to $-f$.

Finally if $0 < \lambda < r$, the diff. $f(a+\lambda t) - f(a)$ has the same sign as $Q(t)$. Hence if $Q(t)$ takes both +ve and -ve values, it follows that f has a saddle point at a

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