Scheme

Course Code 24 PMSMT 1111
Course Title: Diff. Geometry

Max. marks: 75
Time: 3 Hrs

Section A

$10 \times 1 = 10$ marks

1 Curve in R^3 defined by a pair of equation:

$f_1(x,y,z) = C_1$, $f_2(x,y,z) = C_2$ is called level curves.

2. $x = \cos^3 t$, $y = \sin^3 t$ is the parametric representation

3. The parametrization of unit sphere is

$\sigma(\theta, \varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$ latitude θ longitude φ.

σ, $\tilde{\sigma}$ are smooth — $\tilde{\sigma}$ – Rotating σ by π about z-axis then $\pi/2$ about x-axis.

$\sigma_\theta \times \sigma_\varphi = (-\cos^2\theta\cos\varphi, -\cos^2\theta\sin\varphi, -\sin\theta\cos\theta)$

$\|\sigma_\theta \times \sigma_\varphi\| = |\cos\theta|$

If $(\theta, \varphi) \in U$ then $-\pi/2 < \theta < \pi/2$ so $\cos\theta \neq 0$. $\therefore \sigma$ is regular.

Why $\tilde{\sigma}$ is regular

4. Surface comes with a collection of homeos $\sigma : U \to S \cap W$, called surface patches. Its collection is called atlas of S.

5. The principal curvature of a surface patch are the roots of the eqn $\det(F_{II} - \kappa F_I) = 0$ i.e. $\begin{vmatrix} L-\kappa E & M-\kappa F \\ M-\kappa F & N-\kappa G \end{vmatrix} = 0$.

6. $\sigma(u,v) = a + up + vq$

$\sigma_u = p$; $\sigma_v = q$ are const vectors. Further, $\sigma_{uu} = \sigma_{uv} = \sigma_{vv} = 0$

Hence the second fundamental form of a plane is zero.

($\because L\,du^2 + 2M\,du\,dv + N\,dv^2$)

7. A curve γ on a surface S is called if $\ddot{\gamma}(t)$ is zero or perpendicular to the surface at the point $\gamma(t)$. i.e. $\parallel$ to its unit normal vector for all values of the parameter t.

8. $$\frac{d}{dt}\left[E\dot{u}+F\dot{v}\right] = \frac{1}{2}\left[E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2\right]$$

$$\frac{d}{dt}\left[F\dot{u}+G\dot{v}\right] = \frac{1}{2}\left[E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2\right]$$

9. $E=1,\ F=0,\ G=f(u)^2$

$$K = -\frac{1}{\sqrt{G}}\ \frac{\partial^2}{\partial u^2}(\sqrt{G}) = -\frac{\ddot{f}}{f}.$$

10. The Gaussian curvature of a surface is preserved by isometries.

11. Orientable surface is a surface with an atlas having the property that if Φ is the transition map between any two surface patches of the atlas then $\det J(\Phi)>0$ where Φ is defined.

12. $K = k_1k_2$ where k_1 and k_2 are the principal curvatures of a surface patch

Section-B.

$5\times5=25$ marks

Answer any five questions

13. If $\dot{\gamma}(t)=\bar{a}$ for all t, where $\bar{a}$ is a const. vector

we set $\gamma(t)=\int\frac{\partial\gamma}{\partial t}dt = \int\bar{a}\,dt = \bar{a}t+\bar{b}$

where $\bar{b}$ is another const. vector.

If $\bar{a}\neq0$ this is the parametrised eqn of the st. line $\parallel^l$ to $\bar{a}$ and passing thro' the point with p.v. $\bar{b}$. (3 marks)

If $\bar{a}=0$ the image of γ is a single point, the point with position vector $\bar{b}$. (2 marks)

14. $f(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

$\nabla f = (f_x, f_y, f_z) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$

$\|\langle f_x, f_y, f_z\rangle\| = \frac{2}{?}\ x^2+y^2+z^2$

The parametric version is:

$x = a\sin u\cos v$

$y = b\sin u\sin v$

$z = c\cos u$

$\nabla f\neq0$ and its zero only at $(0,0,0)$

But $(0,0,0)$ doesn't satisfy $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$\therefore$ Since $\nabla f\neq0$ at all points on the surface its smooth.

15. $\sigma(u,v) = (u, v, u^2+v^2)$

$\sigma_u = (1, 0, 2u)$ | $\sigma_{uv} = (0,0,0)$

$\sigma_v = (0, 1, 2v)$ | $\sigma_{vv} = (0,0,2)$

$\sigma_{uu} = (0, 0, 2)$

$$\sigma_u \times \sigma_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix}$$

$$= i[-2u] - j[2v] + k[1]$$

$$\|\sigma_u \times \sigma_v\| = \sqrt{4u^2+4v^2+1}$$

$$L = \sigma_{uu} \cdot N = (0,0,2)\frac{(-2u, -2v, 1)}{\sqrt{4u^2+4v^2+1}} = \frac{2}{\sqrt{4u^2+4v^2+1}}$$

$$M = \sigma_{uv} \cdot N = 0$$

$$N = \sigma_{vv} \cdot N = \frac{2}{\sqrt{4u^2+4v^2+1}}$$

$\therefore$ The second fundamental form is $L\,du^2 + 2M\,du\,dv + N\,dv^2$

$$= \frac{2}{\sqrt{4u^2+4v^2+1}}\left[du^2 + dv^2\right].$$

16 It suffice to consider a unit-speed curve γ contained in a patch σ of the surface; let N be the standard normal of σ so that

$$\kappa_g = \ddot{\gamma} \cdot (N \times \dot{\gamma})$$

if $\ddot{\gamma}$ is parallel to N, it is $\perp$lr to $N \times \dot{\gamma}$ so we get $\kappa_g = 0$ —(2marks)

Conversely, Suppose that $\kappa_g = 0$. Then $\ddot{\gamma}$ is $\perp$lr to $N \times \dot{\gamma}$. But then since $\dot{\gamma}$, N and $N \times \dot{\gamma}$ are $\perp$lr unit vectors in $\mathbb{R}^3$, and since $\ddot{\gamma}$ is $\perp$lr to $\dot{\gamma}$, $\ddot{\gamma}$ is $\parallel$l to N. (3marks) . .

17 WKT $e'_u = \alpha e'' + \lambda' N$

$e'_v = \beta e'' + \mu' N$

$e''_u = -\alpha e' + \lambda'' N$

$e''_v = -\beta e' + \mu'' N$

Thus, $e'_u \cdot e''_v - e''_u \cdot e'_v = \lambda'\mu'' - \lambda''\mu'$

[$\because$ e', e'', and N are $\perp^{lr}$ unit vectors.

(2 marks)

$\alpha_v - \beta_u = \frac{\partial}{\partial u}(e' \cdot e''_v) - \frac{\partial}{\partial v}(e' \cdot e''_u)$

$= e'_u \cdot e''_v + e' \cdot e''_{uv} - \{e'_v \cdot e''_u + e' \cdot e''_{uv}\}$

(1 mark)

$= e'_u \cdot e''_v - e'_v \cdot e''_u$

$N_u \times N_v = K\, \sigma_u \times \sigma_v$

$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$, $\|\sigma_u \times \sigma_v\| = \sqrt{EG - F^2}$ (*)

$N_u \times N_v = (LN - M^2)/\sqrt{EG-F^2}\, N$

$\Rightarrow (N_u \times N_v) \cdot N = \frac{LN - M^2}{\sqrt{EG-F^2}}$ ① $\Rightarrow (N_u \times N_v) \cdot N = (N_u \times N_v) \cdot (e' \times e'')$ [$\because N = e' \times e''$

$= \lambda'\mu'' - \lambda''\mu'$ — ②

$N_u \cdot e' = -N \cdot e'_u$; $N_u \cdot e'' = -N \cdot e''_u$; $N_v \cdot e' = -N \cdot e'_v$

$N_v \cdot e'' = -N e''_v$

[$\because N \cdot e' = 0 = N \cdot e''$

from ① & ② we get equality thru' second equality result.

(2 marks)

18 Since $\{\sigma_u, \sigma_v\}$ is a basis of the tgt plane of σ, γ is a geodesic iff $\ddot{\gamma}$ is $\perp^{lr}$ to σ_u and σ_v.

Since $\dot{\gamma} = \dot{u}\sigma_u + \dot{v}\sigma_v$,

$\frac{d}{dt}[\dot{u}\sigma_u + \dot{v}\sigma_v] \cdot \sigma_u = 0$ and $\frac{d}{dt}(\dot{u}\sigma_u + \dot{v}\sigma_v) \cdot \sigma_v = 0$

LHS of ① is equivalent to ~~taking~~   ①   these two eqns are equivalent to geodesic eqn.   ②

$\frac{d}{dt}(\dot{u}\sigma_u + \dot{v}\sigma_v) \cdot \sigma_u - (\dot{u}\sigma_u + \dot{v}\sigma_v) \cdot \frac{d\sigma_u}{dt}$

$$= \frac{d}{dt}\left[E\dot{u}+F\dot{v}\right] - (\dot{u}\,\sigma_u + \dot{v}\,\sigma_v)\cdot(\dot{u}\,\sigma_{uu} + \dot{v}\,\sigma_{uv})$$

$$= \frac{d}{dt}\left[E\dot{u}+F\dot{v}\right] - \left[\dot{u}^2\,\sigma_u\cdot\sigma_{uu} + \dot{u}\dot{v}(\sigma_u\cdot\sigma_{uv}+\sigma_v\cdot\sigma_{uu}) + \dot{v}^2\,\sigma_v\cdot\sigma_{uv}\right]$$

(2 marks)

$$E_u = (\sigma_u\cdot\sigma_u)_u = \sigma_{uu}\sigma_u + \sigma_u\cdot\sigma_{uu} = 2\,\sigma_u\cdot\sigma_{uu}.$$

Don't discard

so,

$$\sigma_u\cdot\sigma_{uu} = \frac{1}{2}E_u.$$

$$\text{lly} \quad \sigma_v\cdot\sigma_{uv} = \frac{1}{2}G_u \quad \text{and} \quad \sigma_u\cdot\sigma_{uv}+\sigma_v\cdot\sigma_{uu} = (\sigma_u\cdot\sigma_v)_u = F_u.$$

$$② \Rightarrow \left(\frac{d}{dt}\left[\dot{u}\sigma_u+\dot{v}\sigma_v\right]\right)\cdot\sigma_u$$

✓

$$= \frac{d}{dt}\left[E\dot{u}+F\dot{v}\right] - \frac{1}{2}\left[E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2\right]$$

Thus, ① is equivalent to first geodesic eqn. and lly ② is equivalent to second geodesic eqn. (5 marks)

19. A map of a region of the earth's surface which doesn't distort distances would be a diffeo. from this region of a sphere to a region in a plane (the map) which multiplied all distances by the same const. factor, say C. We might as well assume that the plane passes thro the origin. Then by composing this map with the map from the plane to itself which takes a point with p.v. $\vec{r}$ to the point with p.v. $C^{-1}\vec{r}$ we would get an isometry between this region of the sphere and some region of a plane. $\Rightarrow$ By the Theorema Egregium, that these regions of the sphere and the plane have the same Gaussian curvature. But WKT a plane has Gaussian curvature zero everywhere, and a sphere has const positive Gaussian curvature everywhere (if the sphere has radius R, the Gaussian curvature is $1/R^2$) So, no such isometry can exist.

Section_c

Answer any four questions. $4\times10=40$ marks

20. Let $\tilde{\gamma}$ (with parameter s) be a unit-speed reparametrization of γ, and let $\frac{d}{ds}$ be denoted by a dash.

By chain rule, $\tilde{\gamma}'\frac{ds}{dt}=\dot{\gamma}$

so,

$$\kappa=\|\tilde{\gamma}''\|=\left\|\frac{d}{ds}\left(\frac{\dot{\gamma}}{ds/dt}\right)\right\|=\left\|\frac{d}{dt}\left(\frac{\dot{\gamma}}{ds/dt}\right)\Big/\frac{ds}{dt}\right\|=\left\|\frac{\ddot{\gamma}\frac{ds}{dt}-\dot{\gamma}\frac{d^2s}{dt^2}}{(ds/dt)^3}\right\|$$

$$\left(\frac{ds}{dt}\right)^2=\|\dot{\gamma}\|^2=\dot{\gamma}\cdot\dot{\gamma} \quad \text{—}(*)$$

Diff wrt t,

$$\frac{ds}{dt}\frac{d^2s}{dt^2}=\dot{\gamma}\cdot\ddot{\gamma} \qquad \text{(Cancelled 2)}$$

$(*)$ $$\kappa=\left\|\frac{\ddot{\gamma}\left(\frac{ds}{dt}\right)^2-\dot{\gamma}\frac{d^2s}{dt^2}\frac{ds}{dt}}{(ds/dt)^4}\right\|=\frac{\|\ddot{\gamma}(\dot{\gamma}\cdot\dot{\gamma})-\dot{\gamma}(\dot{\gamma}\cdot\ddot{\gamma})\|}{\|\dot{\gamma}\|^4} \quad \text{—}(+)$$

(5 marks)

Using vector triple product $a\times(b\times c)=(a\cdot c)b-(a\cdot b)c$

we get $\dot{\gamma}\times(\ddot{\gamma}\times\dot{\gamma})=(\dot{\gamma}\cdot\dot{\gamma})\ddot{\gamma}-(\dot{\gamma}\cdot\ddot{\gamma})\dot{\gamma}$

Further, $\dot{\gamma}$ and $\ddot{\gamma}\times\dot{\gamma}$ are $\perp^{lr}$ vectors.

So, $\|\dot{\gamma}\times(\ddot{\gamma}\times\dot{\gamma})\|=\|\dot{\gamma}\|\,\|\ddot{\gamma}\times\dot{\gamma}\|\sin 90^\circ$

$\therefore (+)\Rightarrow \kappa=\|\dot{\gamma}\times(\ddot{\gamma}\times\dot{\gamma})\|\big/\|\dot{\gamma}\|^4=\|\dot{\gamma}\|\,\|\ddot{\gamma}\times\dot{\gamma}\|\big/\|\dot{\gamma}\|^4$

ie $$\kappa=\frac{\|\ddot{\gamma}\times\dot{\gamma}\|}{\|\dot{\gamma}\|^3}.$$

(5 marks)

21. The patch $\tilde{\sigma}$ is smooth because any composite of smooth maps smooth

As ~~for~~ regularity, let $(u,v)=\Phi(\tilde{u},\tilde{v})$

By chain rule, $$\tilde{\sigma}_{\tilde{u}}=\frac{\partial u}{\partial\tilde{u}}\cdot\sigma_u+\frac{\partial v}{\partial\tilde{u}}\sigma_v$$

$$\tilde{\sigma}_{\tilde{v}}=\frac{\partial u}{\partial\tilde{v}}\sigma_u+\frac{\partial v}{\partial\tilde{v}}\sigma_v$$

$$\tilde{\sigma}_{\tilde{u}}\times\tilde{\sigma}_{\tilde{v}}=\left(\frac{\partial u}{\partial\tilde{u}}\frac{\partial v}{\partial\tilde{v}}-\frac{\partial u}{\partial\tilde{v}}\frac{\partial v}{\partial\tilde{u}}\right)\sigma_u\times\sigma_v. \quad \text{—}(*)$$

~~The scalar on the RHS of this eqn. is the det. of the Jacobian matrix~~

$$~~J(\Phi) = \begin{vmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{vmatrix} \text{ of } \Phi~~$$

(5 marks)

~~If ψ and $\tilde{\psi}$ are two maps between open sets in R^2~~

$$~~J(\tilde{\psi} \circ \psi) = J(\tilde{\psi}) J(\psi)~~$$

~~Taking $\psi = \phi$ and $\tilde{\psi} = \Phi^{-1}$,~~

~~we get $J(\Phi^{-1}) = J(\Phi)^{-1}$.~~

~~In particular, $J(\Phi)$ is invertible, so its det is non-zero~~

(5 marks)

(ii) $\Rightarrow$ $\tilde{\sigma}$ is regular.

2(b). Since the length of any curve can be computed as the sum of the lengths of curves each lying in a single surface patch, we can assume that S_1 and S_2 are covered by single surface patches. Moreover, since f is a diffeo. we can assume that these patches are of the form $\sigma_1 : U \to R^3$ for S_1 and $f \circ \sigma_1 = \sigma_2$ for S_2.

We show that f is an isometry iff σ_1 and σ_2 have the same fundamental form.

Suppose first that σ_1 and σ_2 have the same first fundamental form. If $t \to (u(t), v(t))$ is any curve in U and $\gamma_1(t) = \sigma_1(u(t), v(t))$ and $\gamma_2(t) = \sigma_2(u(t), v(t))$ are the corresponding curves in S_1 and S_2 then f takes γ_1 to γ_2 since

$$f(\gamma_1(t)) = f(\sigma_1(u(t), v(t))) = \sigma_2(u(t), v(t)) = \gamma_2(t)$$

Clearly, $\gamma_1(t)$ and $\gamma_2(t)$ have the same length, since both lengths are found by integrating the expression $\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}$ where $Edu^2 + 2Fdudv + Gdv^2$ is the (common) first fundamental form for σ_1 and σ_2.

(5 marks)

Conversely, suppose f is an isometry. If $t \to (u(t), v(t))$ is any curve in U, defined for $t \in (\alpha, \beta)$ say, the curves $\gamma_1(t) = \sigma_1(u(t), v(t))$

and $\gamma_2(t) = \sigma_2(u(t), v(t))$ have the same length

Hence
$$\int_{t_0}^{t_1} \sqrt{E_1 \dot u^2 + 2F_1 \dot u \dot v + G_1 \dot v^2}\, dt = \int_{t_0}^{t_1} \sqrt{E_2 \dot u^2 + 2F_2 \dot u \dot v + G_2 \dot v^2}\, dt$$

for all $t_0, t_1 \in (\alpha, \beta)$ where

E_1, F_1, G_1 are the coeffts of the first fundamental form of σ_1 and E_2, F_2, G_2 . . . σ_2.

$\Rightarrow$ The two integrands are the same, and hence

$$E_1 \dot u^2 + 2F_1 \dot u \dot v + G_1 \dot v^2 = E_2 \dot u^2 + 2F_2 \dot u \dot v + G_2 \dot v^2 \quad \text{—} \oplus$$

Fix $t_0 \in (\alpha, \beta)$ and let $u_0 = u(t_0)$, $v_0 = v(t_0)$

Apply $\oplus$ for the following three choices of the curve $t \to (u(t), v(t))$ in U:

(i) $u = u_0 + t - t_0$, $v = v_0$ then gives $E_1 = E_2$

(ii) $u = u_0$, $v = v_0 + t - t_0$ then gives $G_1 = G_2$

(iii) $u = u_0 + t - t_0$, $v = v_0 + t - t_0$ then gives

$$E_1 + 2F_1 + G_1 = E_2 + 2F_2 + G_2$$

and hence by (i) & (ii), $F_1 = F_2$. (Smooth)

2 (i) Let t_1 and t_2 be any two $\perp$lr unit tangent vectors to the surface at P. Define ξ_i, η_i and T_i for $i = 1, 2$ as

let $T_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$ $i = 1, 2$
$$t_1 \cdot t_2 = (\xi_1 \sigma_u + \eta_1 \sigma_v) \cdot (\xi_2 \sigma_u + \eta_2 \sigma_v)$$
$$= E \xi_1 \xi_2 + F(\xi_1 \eta_2 + \xi_2 \eta_1) + G \eta_1 \eta_2$$
$$= T_1^t F_1 T_2, \quad F_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

let $A = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix}$

$$A^t F_1 A = \begin{pmatrix} t_1 \cdot t_1 & t_1 \cdot t_2 \\ t_2 \cdot t_1 & t_2 \cdot t_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad [\because t_1 \perp t_2 \text{ unit vectors}$$

Then G_{11} is still (real &) symmetric because $G_{11}^t = A^t F_1^t (A^t)^t$
$= A^t F_{11} A = G_{11}$

Then there is an orthogonal matrix B s.t.

$$B^t G_{11} B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ for some real nos. } \lambda_1, \lambda_2$$

let $C = AB$ Then $C^t F_1 C = B^t (A^t F_1 A) B = B^t B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ — ①

Also, since B is orthogonal

$$C^t F_{11} C = B^t (A^t F_{11} A) B = B^t G_{11} B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ — ②}$$

Now C is invertible so,

$$\det (F_{11} - k F_1) = 0 \text{ iff } \det [C^t (F_{11} - k F_1) C]$$

$$\therefore \det (F_{11} - k F_1) = 0 \text{ iff } \det \left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0$$

Hence the principal curvature are the roots of

$$\begin{vmatrix} \lambda_1 - k & 0 \\ 0 & \lambda_2 - k \end{vmatrix} = 0$$

(4 marks)

i.e. λ_1 and λ_2

(ii) Suppose that the principal curvatures are equal to k, say.

Then $\lambda_1 = \lambda_2 = k$ and ① & ② give

$$C^t F_1 C = I, \quad C^t F_{11} C = kI$$

$$C^t (F_{11} - k F_1) C = 0$$

$$\therefore F_{11} - k F_1 = 0.$$

because C and C^t are invertible. Then if T is any 2×1 column matrix

$$(F_{11} - k F_1) T = 0$$

→ Every tgt vector to S at $\underline{P}$ is a principal — (3 marks).

(iii) let $t_i = (\xi_i \sigma_u + \eta_i \sigma_v)$ $T_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$ for $i = 1, 2$

Then $t_1 \cdot t_2 = T_1^t F_1 T_2$ and $F_{11} T_1 = k_1 F_1 T_1$, $F_{11} T_2 = k_2 F_1 T_2$

$\Rightarrow$ $T_2^t F_{11} T_1 = k_1 (t_1 \cdot t_2)$, $T_1^t F_{11} T_2 = k_2 (t_1 \cdot t_2)$ — ③

But $T_1^t F_{11} T_2$ is a 1×1 matrix, it is equal to its transpose

$$T_1^t F_{11} T_2 = (T_1^t F_{11} T_2)^t = T_2^t F_{11}^t T_1 = T_2^t F_{11} T_1 \quad [\because F_{11} \text{ is Symmetric}$$

③ $\Rightarrow$ $k_1 (t_1 \cdot t_2) = k_2 (t_1 \cdot t_2)$

So if $k_1 \neq k_2$, $t_1 \cdot t_2 = 0 \Rightarrow t_1 \perp t_2$ (3 marks)

2. If $f(\tau, t)$ is smooth $\frac{d}{d\tau} \int f(\tau, t) \, dt = \int \frac{\partial f}{\partial \tau} \, dt$

Then, $\frac{d}{d\tau} L(\tau) = \frac{d}{d\tau} \int_a^b \| \dot{\gamma} \| \, dt = \frac{d}{d\tau} \int_a^b \sqrt{E \dot{u}^2 + 2F \dot{u}\dot{v} + G \dot{v}^2} \, dt$

$$= \int_a^b \frac{\partial}{\partial \tau} (g(\tau, t)^{1/2}) \, dt = \frac{1}{2} \int_a^b g(\tau, t)^{-1/2} \frac{\partial g}{\partial \tau} \, dt$$

where $g(\tau,t) = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$

$$\frac{\partial g}{\partial \tau} = (E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)\frac{\partial u}{\partial \tau} + (E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)\frac{\partial v}{\partial \tau}$$
$$+ 2(E\dot{u} + F\dot{v})\frac{\partial^2 u}{\partial \tau \partial t} + 2(F\dot{u} + G\dot{v})\frac{\partial^2 v}{\partial \tau \partial t}$$

$$\int_a^b g^{-1/2}\left\{(E\dot{u} + F\dot{v})\frac{\partial^2 u}{\partial \tau \partial t} + (F\dot{u} + G\dot{v})\frac{\partial^2 v}{\partial \tau \partial t}\right\} dt$$

$$= g^{-1/2}\left\{(E\dot{u} + F\dot{v})\frac{\partial u}{\partial \tau} + (F\dot{u} + G\dot{v})\frac{\partial v}{\partial \tau}\right\}\Big|_{t=a}^{b}$$

$$- \int_a^b \left(\frac{\partial}{\partial t}\left\{g^{-1/2}(E\dot{u} + F\dot{v})\right\}\frac{\partial u}{\partial \tau} + \frac{\partial}{\partial t}\left\{g^{-1/2}(F\dot{u} + G\dot{v})\right\}\frac{\partial v}{\partial t}\right) dt \quad (*)$$

since $\gamma^\tau(a)$ & $\gamma^\tau(b)$ are indep of τ [being equal to p and q resp'ly

we've $\frac{\partial \gamma^\tau}{\partial \tau} = 0$ when $t = a$ or b

since $\frac{\partial \gamma^\tau}{\partial \tau} = \frac{\partial u}{\partial \tau}\sigma_u + \frac{\partial v}{\partial \tau}\sigma_v$ we've $\frac{\partial u}{\partial \tau} = 0, \frac{\partial v}{\partial \tau} = 0$. When $t = a$ or b.

RHS of $(*)$ is zero.

Thus, $$\frac{d}{d\tau} L(\gamma) = \int_a^b \left(U\frac{\partial u}{\partial \tau} + V\frac{\partial v}{\partial \tau}\right) dt$$

where
$$U(\tau,t) = \frac{1}{2} g^{-1/2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2) - \frac{d}{dt}\left(g^{-1/2}(E\dot{u} + F\dot{v})\right)$$
$$V(\tau,t) = \frac{1}{2} g^{-1/2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2) - \frac{d}{dt}\left\{g^{-1/2}(F\dot{u} + G\dot{v})\right\} \quad (+)$$

now $\gamma^0 = \gamma$ is unit-speed. so since $\|\dot{\gamma}^\tau\| = g(\tau,t)$

$g(\tau,t) = 1 \ \forall t$ when $\tau = 0$.

Comparing $(+)$ with geodesic eqns, if γ is geodesic then $U = V = 0$ when $\tau = 0$

so $\frac{d}{d\tau}\left(L(\gamma)\right|_{\tau=0} = 0$ when $\tau = 0$ (Geodesic)

for converse

if $$\int_a^b \left(U\frac{\partial u}{\partial \tau} + V\frac{\partial v}{\partial \tau}\right) dt = 0 \text{ when } \tau = 0 \text{ for all families of curves } \gamma^\tau \quad (*)$$

then $U = V = 0$ when $\tau = 0$.

Assume $(*)$ and suppose $U \neq 0$ when $\tau = 0$ get a contradiction

(Geodesic)

Proof 10.1

Combining Eqs. (3) and (4), we get

$$K = \frac{\alpha_v - \beta_u}{(EG - F^2)^{1/2}}, \tag{7}$$

so to prove the theorem it suffices to show that, *for a suitable choice of* $\{\mathbf{e}', \mathbf{e}''\}$, *the scalars* α *and* β *depend only on* E, F *and* G. We shall construct $\{\mathbf{e}', \mathbf{e}''\}$ by applying the Gram–Schmidt process to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ of the tangent plane, and will then show that they have the desired property.

So we first define

$$\mathbf{e}' = \frac{\boldsymbol{\sigma}_u}{\| \boldsymbol{\sigma}_u \|} = \epsilon \boldsymbol{\sigma}_u,$$

where $\epsilon = E^{-1/2}$. Now we look for a vector $\mathbf{e}'' = \gamma \boldsymbol{\sigma}_u + \delta \boldsymbol{\sigma}_v$, for some scalars γ, δ, such that $\mathbf{e}''$ is a unit vector perpendicular to $\mathbf{e}'$. These conditions give

$$E^{-1/2}(\gamma E + \delta F) = 0, \quad \gamma^2 E + 2\gamma\delta F + \delta^2 G = 1.$$

The first equation gives $\gamma = -\delta F/E$, and substituting in the second equation then gives

$$\delta^2 \left(\frac{F^2}{E} - 2\frac{F^2}{E} + G \right) = 1,$$

$$\delta = \frac{E^{1/2}}{(EG - F^2)^{1/2}}, \quad \gamma = -\frac{F E^{-1/2}}{(EG - F^2)^{1/2}}, \quad \epsilon = E^{-1/2} \tag{8}$$

(We could change the sign of δ, and hence also that of γ, but it would make no difference in the end.) Thus,

$$\mathbf{e}' = \epsilon \boldsymbol{\sigma}_u, \quad \mathbf{e}'' = \gamma \boldsymbol{\sigma}_u + \delta \boldsymbol{\sigma}_v, \tag{9}$$

where γ, δ and ϵ depend only on E, F and G.

$$
\begin{aligned}
&= \frac{\epsilon_u}{\epsilon}\mathbf{e}'.\mathbf{e}'' + \frac{1}{2}\epsilon\gamma(\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_u)_u + \epsilon\delta((\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_v)_u - \boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{uv}) \\
&= \frac{1}{2}\epsilon\gamma E_u + \epsilon\delta(F_u - \frac{1}{2}E_v) \quad (\text{since } \mathbf{e}'.\mathbf{e}'' = 0), \qquad (10)
\end{aligned}
$$

which does indeed depend only on E, F and G (because the same is true for γ, δ and ϵ). And finally,

$$
\begin{aligned}
\beta &= \mathbf{e}'_v.\mathbf{e}'' \\
&= (\epsilon_v\boldsymbol{\sigma}_u + \epsilon\boldsymbol{\sigma}_{uv}).(\gamma\boldsymbol{\sigma}_u + \delta\boldsymbol{\sigma}_v) \\
&= \frac{\epsilon_v}{\epsilon}\mathbf{e}'.\mathbf{e}'' + \epsilon\gamma\boldsymbol{\sigma}_{uv}.\boldsymbol{\sigma}_u + \epsilon\delta\boldsymbol{\sigma}_{uv}.\boldsymbol{\sigma}_v \\
&= -\frac{1}{2}\epsilon\gamma E_v + \frac{1}{2}\epsilon\delta G_u, \qquad (11)
\end{aligned}
$$

which also depends only on E, F and G.

This completes the proof of Gauss's Theorem. □

By substituting the actual values of γ, δ and ϵ into these formulas for α and β, and then using Eq. (7), we get an *explicit* formula for K in terms of E, F and G. Here is the result:

Let us compute the curvature of the helix using the formula in Proposition 2.1. Denoting $d/d\theta$ by a dot, we have

$$\dot{\gamma}(\theta) = (-a\sin\theta, a\cos\theta, b),$$
$$\therefore \quad \| \dot{\gamma}(\theta) \| = \sqrt{a^2 + b^2}.$$

This shows that $\dot{\gamma}(\theta)$ is never zero, so γ is regular (unless $a = b = 0$, in which case the image of the helix is a single point). Hence, the formula in Proposition 2.1 applies, and we have

$$\ddot{\gamma} = (-a\cos\theta, -a\sin\theta, 0),$$
$$\therefore \quad \ddot{\gamma} \times \dot{\gamma} = (-ab\sin\theta, ab\cos\theta, -a^2),$$
$$\therefore \quad \kappa = \frac{\| (-ab\sin\theta, ab\cos\theta, -a^2) \|}{\| (-a\sin\theta, a\cos\theta, b) \|^3} = \frac{(a^2b^2 + a^4)^{1/2}}{(a^2 + b^2)^{3/2}} = \frac{|a|}{a^2 + b^2}. \tag{3}$$

Thus, the curvature of the helix is constant.

Let us examine some limiting cases to see if this result agrees with what we already know. First, suppose that $b = 0$ (but $a \neq 0$). Then, the helix is simply a circle in the xy-plane of radius $|a|$, so by the calculation following Definition 1.1 its curvature is $1/|a|$. On the other hand, the formula (3) gives the curvature as

$$\frac{|a|}{a^2 + 0^2} = \frac{|a|}{a^2} = \frac{|a|}{|a|^2} = \frac{1}{|a|}.$$

Next, suppose that $a = 0$ (but $b \neq 0$). Then, the image of the helix is just the z-axis, a straight line, so the curvature is zero. And (3) gives zero when $a = 0$ too.