Functional Analysis

Answer Key.

Part-A (10×1=10 Marks)

1. Banach Space:

A Banach space is a complete normed linear space.

2. Open Mapping theorem.

If B and B' are Banach spaces, and if T is a continuous linear transformation of B onto B', then T is an open mapping.

3. Hilbert Space:

A complex inner product space.

4. Perpendicular projection:

A projection on H whose range the null space are orthogonal is sometimes called a perpendicular projection.

5. Eigen vector:

$Tx = \lambda x$.

6. Spectral Resolution of T.

$T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots \lambda_m P_m$ exists.

7. Banach Algebra.

(1) $\|xy\| \leq \|x\| \, \|y\|$ (2) $\|I\| = 1$

8 Convolution :

If two functions f & g in $L_1(G)$ are given, then their product $f*g$ called their Convolution.

9. Maximal ideal space. $(f*g)\, g_k = \sum_{g_i g_j = g_k} f(g_i)\, g(g_j)$

We call the topological space M the space of maximal ideals or the maximal ideal space.

10. Multiplicative Functional

An element of the conjugate space A^* — which is non zero and satisfies $f(xy) = f(x)\, f(y)$

11. N in N^{**}

The map $F_x : N \to N^{**}$ defined by

$F_x(f) = f(x) \quad \forall f \in N^*$.

12. Spectral radius :

$r(x) = \sup \{ |\lambda| : \lambda \in \sigma(x) \}$.

Part B $(5 \times 5 = 25)$

13. Using Holder's inequality.

$|f+g|^p \leq (|f| + |g|)^p$.

$\int$ both sides.

$\left(\int |f+g|^p\right)^{1/p} \leq \left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}$

Hence Minkowski's inequality holds.

14. Schwarz Inequality.

If x and y are any two vectors in a Hilbert Space then $|(x,y)| \leq \|x\| \, \|y\|$.

Proof:

When $y=0$, the result is clear.

$y \neq 0 \Rightarrow |(x, y/\|y\|)| \leq \|x\|$.

if $\|y\| = 1$ then $|(x,y)| \leq \|x\| \ \forall x$.

To proving that if $\|y\| = 1$ then we have $|(x,y)| \leq \|x\| \ \forall x$.

$\Rightarrow 0 \leq \|x - (x,y)y\|^2 = \|x\|^2 - |(x,y)|^2$.

then reach the proof.

15. If T is Normal, then the M_i's span H.

-: By the thm $M = M_1 + M_2 + \cdots + M_m$ is a closed linear subspace H, and associated projection is $P = P_1 + P_2 \cdots + P_m$.

Where $TP_i = P_i T$ for each $i \Rightarrow TP = PT$

If $M^\perp \neq \{0\}$ no eigenvector and value.

If $M^\perp = \{0\}$ so $M = H$ and the M_i's span H.

16. If $1-xr$ is regular, then $1-rx$ is also regular.

Proof: Assume $1-xr$ is regular.

$$s = (1-xr)^{-1}.$$

$$(1-rx)(1+rsx) = (1+rsx)(1-rx) = 1.$$

$$s = (1-xr)^{-1} = 1 + xr + (xr)^2 + \cdots$$

$$(1-rx)^{-1} = 1 + rsx.$$

Thus finally arrived $1-rx$ is regular

17. If A is self adjoint, the $\hat{A}$ is dense is $\mathcal{C}(\mathfrak{M})$

proof! Let A be self-adjoint Banach algebra and $\hat{A}$ its Gelfand transform.

By Stone Weierstrass thm, the image $\hat{A}$ seperates points and contains constants.

$\therefore$ $\hat{A} = \mathcal{C}(\mathfrak{M})$, so $\hat{A}$ is dense in $\mathcal{C}(\mathfrak{M}$

18. Show that $H = M \oplus M^{\perp}$.

Proof! For every $x \in H$ $\exists$ a unique $m \in M$ s.t $x = m + n$, $n \in M^{\perp}$.

Thus $H = M \oplus M^{\perp}$

19. $\sigma(x)$ is non-empty.

Resolvent equation shows that

$$\frac{f(\lambda)-f(\mu)}{\lambda-\mu} = f(x(\lambda)\, x(\mu)),$$

$$\Rightarrow \lim_{\lambda\to\mu} \frac{f(\lambda)-f(\mu)}{\lambda-\mu} = f(x(\mu)^2).$$

So $f(\lambda)$ has a derivative at each point of $\rho(x)$ furthere $|f(\lambda)| \le \|f\|\,\|x(\lambda)\|$

Assume $\sigma(x) = \varphi$ then $(\lambda I - x)^{-1}$ exists.

This leads to an entire bounded resolvent function, contradicting Liouville's thm, Hence $\sigma(x) \neq 0$.

Section - C

20. Hahn - Banach thm: Statement:

Let M be a linear subspace of a NLS N, and let f be a functional defined on M. Then f can be extended to a functional fo defind on the whole space N s.t $\|f_0\| = \|f\|$.

21. **Bessell's inequality!** **Statement:**

If $\{e_i\}$ is an orthonormal set in a Hibert space H, then $\Sigma |(x, e_i)|^2 \leq ||x||^2$ for every vector x in H.

22. **Operator Non-Singularity and Matrix Representation.**

Proof!

If T is a non singular operator and T exists and is linear.

Hence $TT^{-1} = I$.

thus $[\alpha_{ij}]$ is non-singular.

Conversely! If α_{ij} is nonsingular, then the linear transformation defined by it has an inverse, implying T^{-1} exists.

Hence $[T^{-1}] = [\alpha_{ij}]^{-1}$

23. **Spectral Radius formule!**

Proof!

Let $r(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$.

From properties of resolvent and submultiplicity.

$$\|x^n\|^{1/n} \le r(x) + \varepsilon \quad (n \text{ large})$$

Conversely, for any $|\lambda| > r(x)$, $(\lambda I - x)^{-1}$ exists

$$\Rightarrow \|x^n\|^{1/n} \ge r(x)$$

Hence the limit exists and equals $r(x)$.

24. **Gelfand - Neumark Representation thm:**

Statement:

If A is a Commutative B^*-algebra, then the Gelfand mapping $x \to \hat{x}$ is an isometric $*$-isomorphism of A onto the Commutative B^*-algebra $C(\mathcal{M})$.

25. **Uniform Boundedness thm:**

Let B be a Banach space and N a normed linear space. If $\{T_i\}$ is a non-empty set of Continuous linear transformations of B into N with the property that $\{T_i(x)\}$ is a bounded subset of N for each vector x in B, then $\{\|T_i\|\}$ is a bounded set of numbers; that is $\{T_i\}$ is bounded as a subset of $\mathcal{B}(B,N)$.