24PMSMT112 Functional Analysis

Answer key

Part-A

1. A complete normed Linear space. R^n.
2. The set of continuous Linear Transformation from $N \to R$ or C. ie $\mathcal{B}(N,R)$ or $\mathcal{B}(N,C)$
3. $\|f\| = \sup |f(x)|$. ($f_n \in C_n(X)$).
4. Prove $\|e_n\| = 1$ and $\langle e_n, e_m \rangle = 0$.
5. $f_y(\alpha x) = \langle \alpha x, y \rangle = \alpha \langle x, y \rangle = \alpha f_y(x)$.
   $f_y(x_1 + x_2) = \langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle = f_y(x_1) + f_y(x_2)$
6. $\langle x, \alpha y + \beta z \rangle = \overline{\langle \alpha y + \beta z, x \rangle} = \bar{\alpha} \overline{\langle y, x \rangle} + \bar{\beta} \overline{\langle z, x \rangle} = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$.
7. $\exists$ a scalar λ st $Ax = \lambda x$, then λ is called eigen value, The nonzero column vector x is called eigen vector.
8. If $\exists$ a sequence $\{z_n\}$ in a Banach Algebra A, st $\|z_n\| = 1$ and either $z z_n \to 0$ or $z_n z \to 0$.
9. $\overline{(1 - rx)(1 + rx)} = (1 + r s(x))(1 - r s(x)) = 1$.
10. $\lambda \neq 0 \in \mathbb{C}$ and $\lambda_1, \lambda_2 \ldots \lambda_n$ its distinct nth roots, so that $x^n - \lambda I = (x - \lambda_1 I) \ldots (x - \lambda_n I) \Rightarrow x^n - \lambda I$ is singular $\Leftrightarrow x - \lambda_i I$ is singular for atleast one i.
11. If $\exists$ a mapping $x \to x^*$ of A into itself with
   1) $(x + y)^* = x^* + y^*$ 2) $(\alpha x)^* = \bar{\alpha} x^*$ 3) $(xy)^* = y^* x^*$ 4) $x^{**} = x$.
12. $\|x^2\| = \|(x)^2\| = \|x'\|^2 = \|x\|^2$ (using spectral radius formula)

Part-B

13. Open mapping theorem: If B and B' are Banach space and if T is a continuous Linear Transformation of B onto B'. Then T is a open mapping.

Prove : The Image of each open sphere centered at origin in B contains an open sphere in B'. and Prove T(G) is open in B', if G is open in B.

14. Let $T = A_1 + i A_2$ and $T^* = A_1 - i A_2$.
Prove $A_1 A_2 = A_2 A_1$ then $TT^* = T^* T$.
Conversely, if $TT^* = T^* T$ then $A_1 A_2 = A_2 A_1 = A_2 A_1 - A_1 A_2$
so $2 A_1 A_2 = 2 A_2 A_1 \Rightarrow A_1 A_2 = A_2 A_1$

15. Bessel's Inequality

If $\{e_i\}$ is an orthonormal set in a Hilbert space H, then $\sum |\langle x, e_i\rangle|^2 \le \|x\|^2$.

16. Let $Te_j = \sum_{i=1}^{n} \alpha_{ij} e_i$ compute $T\beta_j$ in two different way and prove $[\alpha_{ij}][B_{ij}] = [\alpha_{ij}][\alpha_{ij}]$
& $[T]_B = [A]_B^{-1} [T]_B [A]_B$.

17. Assume $1 - xy$ is regular. Its Inverse is $s = (1-xy)^{-1}$.
Prove $(1-yx)(1+syx) = (1+ysx)(1-yx) = 1$
$\Rightarrow (1-yx)$ is regular.

Let $x_0 \in G$ and x be another element of G st $\|x - x_0\| < 1/(2\|x_0^{-1}\|)$

18 Prove
$$\|x^{-1} - x_0^{-1}\| = \frac{\|x_0^{-1}\| \, \|1 - x_0^{-1}x\|}{1 - \|1 - x_0^{-1}x\|} < 2\|x_0^{-1}\| \, \|1 - x_0^{-1}x\| \le 2\|x_0^{-1}\|^2 \|x - x_0\|$$

19. Let $x_0 \in A$ and not in M. If x is an arbitrary elements of A $\Rightarrow x = m + \beta x_0$.
Now, $f_1(x) = f_1(m) + \beta f_1(x_0) = \beta \beta_1 f(x_0) = [f_1(x_0)/f_2(x_0)] f_2(x)$
So $f_1 = \alpha f_2$ with $\alpha = f_1(x_0)/f_2(x_0)$, ~~Prove $\alpha = 1$~~
Prove $\alpha = 1$.

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Part-C

20 $\|x+M\| = \{\|x+m\| : m \in M\}$

1) Prove $\frac{N}{M}$ is a Normed Linear space.

2) Prove cauchy seq (x_n) in M, converged in N/M.

21) Hahn Banach thm : Let M be a linear subspace of a normed Linear space N and let f be a functional defined on M. Then f can be extended to a functional f_0 defined on the whole space N st $\|f_0\| = \|f\|$.

Lemma : Let M be a linear subspace of a Normed Linear space N and let f be a functional defined on M. If x_0 is a vector not in M and if $M_0 = M + [x_0]$ is a ls spanned by M and x_0, then f can be extended to a functional f_0 on M_0 st $\|f_0\| = \|f\|$

22) Prove if such y exist, then y is unique.

Show that if α is suitably choosen, then the vector $y = \alpha y_0$ meets the requirement.

choose $\alpha = f(\bar{y_0})/\|y_0\|^2$ & prove $f(x) = \langle x, y \rangle$.

23) The following 3 statements are equivalent. Let T be an arbitrary operator on H. Let $M_1, M_2 \ldots M_m$ be corr. ~~eigenvalue~~ eigenspaces and $P_1, \ldots P_m$ be Projections on these eigen spaces

1) M_i's are pairwise orthogonal and span H.

2) The P_i's are pairwis orthogonal, $I = \sum_{i=1}^{n} P_i$ & $T = \sum \lambda_i P_i$

3) T is normal

24. $r(x) = \lim \|x^n\|^{1/n}$. Prove 1) $r(x^n) = (r(x))^n$.

2) Prove if a is any real number st $r(x) < a$, then $\|x^n\|^{1/n} \leq a \ \forall a$. but a finite numbers of 'a'.

Gelfand - Neumark theorem

25. Prove if A is selfadjoint and if $\|x^2\| = \|x\|^2 \ \forall x$ then Gelfand mapping $x \to \hat{x}$ is an ~~isomorphic~~ isometric isomorphism of A on $C(M)$ and

Show that $\hat{x^*}(M) = \overline{\hat{x}(M)}$ for all $x \in A$.