PROBABILITY THEORY

PART- A (10 × 1 = 10)

1. Conditional Probability $P(A|B) = \frac{P(AB)}{P(B)}$

2. Non-increasing sequence of limits $A_n \supset A_{n+1}$

3. Distribution function of random variable (X,Y).$F(x, y) = P(X < x, Y < y)$

4. expected value $E(X) = \sum_k p_k x_k = \int f(x)p(x)dx$

5. $E(X^2)$ = 4x0.2+16x0.8=13.6

6. Variance $D^2(X) = E(X^2) - \{E(X)]$^2}

7. characteristic function $\phi(t) = E(e^{itx})$

8. probability generating function. $\psi(t) = \sum_k p_k s^k$

9. Polya distribution.

$$= \binom{n}{k} \frac{p(p+\alpha)\ldots[p+(k-1)\alpha]q(q+\alpha)\ldots[q+(n-k-1)\alpha]}{1(1+\alpha)\ldots[1+(n-1)\alpha]}.$$

10. Normal distribution.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right),$$

11. Stochastically convergent to zero.

The sequence $\{X_n\}$ of random variables is called *stochastically convergent*[1] *to zero* if for every $\varepsilon > 0$ the relation

$$\lim_{n\to\infty} P(|X_n| > \varepsilon) = 0$$

is satisfied.

12. Convergent sequence $\{F_n(x)\}$ of distribution function of random variables $\{X_n\}$.

The sequence $\{F_n(x)\}$ of distribution functions of the random variables $\{X_n\}$ is called *convergent*, if there exists a distribution function $F(x)$ such that, at every continuity point of $F(x)$, the relation

$$\lim_{n\to\infty} F_n(x) = F(x)$$

is satisfied. The distribution function $F(x)$ is called the *limit distribution function*.

PART - B (5 × 5 = 25)

13. Proof :

$$P(A \cup B) = P(A) + P(B - AB),$$
$$P(B) = P(AB) + P(B - AB).$$

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

14. Baye's theorem Proof :

Using the multiplication rule of probability,

$$P(A_iB) = P(A_i)P(B|A_i) \quad \dots (1)$$

By theorem of absolute probability, $B = A_1B + A_2B + \cdots + A_nB$

$$P(B) = P(A_1\mathrm{B}) + P(A_2\mathrm{B}) + \cdots + \mathrm{P}(A_n\mathrm{B}) = \sum_{i=1}^{n} P(A_iB)$$

$$= \sum_{i=1}^{n} P(A_i)P(B|A_i) \dots (2)$$

According to the conditional probability formula,

$$P(A_i|B) = \frac{P(A_iB)}{P(B)} \quad \dots (3)$$

Using equations (1) and (2) in equation (3), we get

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}$$

15. mean and variance of binomial distribution.

$$E(X) = \sum_{r=0}^{n} r \frac{n!}{r!\,(n-r)!} p^r (1-p)^{n-r}$$

$$= \sum_{r=1}^{n} r \frac{n!}{r!\,(n-r)!} p^r (1-p)^{n-r} = np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!\,(n-r)!} p^{r-1}(1-p)^{n-r}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} p^k (1-p)^{n-1-k} = np[p + (1-p)]^{n-1} = np.$$

$$E(X^2) = \sum_{r=0}^{n} r^2 \binom{n}{r} p^r (1-p)^{n-r} = pn(q + pn),$$

The mean = np and variance = npq

16. coefficient of correlation satisfies double inequality $-1 \le \rho \le 1$.

$$\begin{aligned} &E\{[t(X - m_{10}) + u(Y - m_{01})]^2\} \\ &= t^2E[(X - m_{10})^2] + 2tuE[(X - m_{10})(Y - m_{01})] \\ &\quad + u^2E[(Y - m_{01})^2] \\ &= t^2\sigma_1^2 + 2tu\mu_{11} + u^2\sigma_2^2. \end{aligned}$$

Since the left-hand side always non-negative, we must have

$$\mu_{11}^2 - \sigma_1^2\sigma_2^2 \leqslant 0, \qquad \text{or} \quad -\sigma_1\sigma_2 \leqslant \mu_{11} \leqslant \sigma_1\sigma_2,$$

and hence

$$-1 \leqslant \frac{\mu_{11}}{\sigma_1\sigma_2} \leqslant 1.$$

17. probability generating function of binomial distribution.

$$p_k = \binom{n}{k} p^k(1 - p)^{n-k} \qquad (k = 0, 1, \ldots, n).$$

$$\psi(s) = \sum_{k=0}^{n} \binom{n}{k} (ps)^k(1 - p)^{n-k} = (ps + q)^n.$$

18. Characteristic function of Gamma distribution.

$$\phi(t) = \int_{-\infty}^{+\infty} e^{itx} f(x)\, dx = \frac{b^p}{\Gamma(p)} \int_0^{+\infty} x^{p-1} e^{-(b-it)x}\, dx.$$

$$= \frac{b^p}{\Gamma(p)} \cdot \frac{\Gamma(p)}{(b - it)^p} = \frac{1}{(1 - it/b)^p}.$$

19. Borel – Cantelli lemma.

$$A = \bigcap_{r=1}^{\infty} \bigcup_{n=r}^{\infty} A_n.$$

$$A \subset \bigcup_{n=r}^{\infty} A_n, \qquad P(A) \leqslant P\left(\bigcup_{n=r}^{\infty} A_n\right) \leqslant \sum_{n=r}^{\infty} P(A_n).$$

$$P(A) = 0$$

$$\bar{A} = \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} \bar{A}_n.$$

$$1 - P(A) = P(\bar{A}) = P\left(\bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} \bar{A}_n\right)$$

$$\leqslant \sum_{r=1}^{\infty} P\left(\bigcap_{n=r}^{\infty} \bar{A}_n\right) = \sum_{r=1}^{\infty} \prod_{n=r}^{\infty} [1 - P(A_n)].$$

$$P(A) = 1$$

SECTION – C (4 X 10 = 40)

20. $Z = XY$ Let $X = X$, $Z = XY$

$$J = \frac{\partial(x,y)}{\partial(x,z)} = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{-z}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{1}{x}$$

The joint density of (X,Z) $= f(x,y)|J| = f\left(x, \frac{z}{x}\right)\frac{1}{|x|}$

The Marginal density of (X,Z) is

$$\psi(z) = \int_{-\infty}^{+\infty} f(x, z - x)\, dx.$$

The distribution function of Z is

$$F(z) = \int_{-\infty}^{z} \left[\int_{-\infty}^{+\infty} f\left(x, \frac{z}{x}\right)\frac{1}{|x|}\, dx\right] dz.$$

21. Lapunov's inequality:

Proof. Suppose that the random variable X is of the continuous type. Let u and v be two arbitrary real numbers. Consider the non-negative expression

$$\int_{-\infty}^{+\infty} [u\,|x|^{(k-1)/2} + v(|x|^{(k+1)/2}]^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} [u^2\,|x|^{k-1} + 2uv\,|x|^k + v^2\,|x|^{k+1}] f(x)\, dx$$

$$= \beta_{k-1}u^2 + 2\beta_k uv + \beta_{k+1}v^2.$$

Since the quadratic form on the right-hand side does not change sign, the inequality

$$\beta_k^2 \leqslant \beta_{k-1}\beta_{k+1}$$

holds. Raising both sides to power k, we obtain

$$\beta_k^{2k} \leqslant \beta_{k-1}^k \beta_{k+1}^k.$$

$Put\ K = 1,2, \dots n-1$

$$\beta_k^{\frac{1}{k}} \leq \beta_{k+1}^{\frac{1}{k+1}}$$

22. (a)

We first show that the random variables X and Y are dependent. The marginal distributions in the domains $|x| \leqslant 1$ and $|y| \leqslant 1$ are, respectively, of the form

$$f_1(x) = \int_{-1}^{+1} \tfrac{1}{4}[1 + xy(x^2 - y^2)]\, dy = \tfrac{1}{4}(y + \tfrac{1}{2}x^3y^2 - \tfrac{1}{4}xy^4)_{-1}^{+1} = \tfrac{1}{2},$$

$$f_2(y) = \int_{-1}^{+1} \tfrac{1}{4}[1 + xy(x^2 - y^2)]\, dx = \tfrac{1}{4}(x + \tfrac{1}{4}x^4y - \tfrac{1}{2}x^2y^3)_{-1}^{+1} = \tfrac{1}{2}.$$

We then obtain $f_1(x)f_2(y) = \frac{1}{4} \neq f(x, y)$; hence the random variables X and Y are not independent.

(b)

$$\phi_1(t) = \frac{1}{2}\int_{-1}^{+1} e^{itx}\, dx = \frac{1}{2}\left[\frac{e^{itx}}{it}\right]_{-1}^{+1} = \frac{e^{it} - e^{-it}}{2it} = \frac{\sin t}{t}.$$

23. Poisson distribution

$$P(X = r) = \frac{\lambda^r}{r!} e^{-\lambda},$$

$$\phi(t) = e^{\lambda(e^{it}-1)}.$$

$$m_1 = \lambda, \quad m_2 = \lambda(\lambda + 1), \quad \mu_2 = \lambda.$$

24. Moments of Beta distribution

$$m_k = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\int_0^1 x^{p+k-1}(1-x)^{q-1}\,dx = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\cdot B(p+k,q)$$

$$= \frac{\Gamma(p+q)\Gamma(p+k)}{\Gamma(p)\Gamma(p+q+k)} = \frac{p(p+1)\ldots(p+k-1)}{(p+q)(p+q+1)\ldots(p+q+k-1)}.$$

25. Lindeberg-Levy theorem.

Theorem *If $X_1, X_2, \ldots$ are independent random variables with the same distribution, whose standard deviation $\sigma \neq 0$ exists, then the sequence $\{F_n(z)\}$ of distribution functions of the random variables Z_n, given by*

$$Y_n = X_1 + X_2 + \ldots + X_n. \qquad Z_n = \frac{Y_n - mn}{\sigma\sqrt{n}}.$$

Then, $$\lim_{n\to\infty} F_n(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-z^2/2}\,dz.$$

$$E(X_k - m) = 0 \qquad \text{and} \qquad D^2(X_k - m) = \sigma^2.$$

$$\phi_x(t) = 1 - \tfrac{1}{2}\sigma^2 t^2 + o(t^2).$$

$$\phi_z(t) = \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n,$$

Let

$$u = -\frac{t^2}{2n} + o\left(\frac{t^2}{n}\right).$$

We obtain

$$\log \phi_z(t) = n\log(1+u) = n\left[-\frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right] = -\frac{t^2}{2} + no\left(\frac{t^2}{n}\right).$$

we obtain $\lim_{n\to\infty} \log \phi_z(t) = -t^2/2$. Hence

$$\lim_{n\to\infty} \phi_z(t) = e^{-t^2/2}.$$