

ANNA ADARSH COLLEGE FOR WOMEN (AUTONOMOUS)

Answer Key

Mathematical Programming

Part – A

1.	There are always certain limitations (or constraints) on the use of resources, such as: labour, machine, raw material, space, money, etc., that limit the degree to which an objective can be achieved. Such constraints must be expressed as linear equalities or inequalities in terms of decision variables. The solution of an LP model must satisfy these constraints.
2.	The initial solution obtained by any of the three methods must satisfy the following conditions: (i) The solution must be feasible, i.e. it must satisfy all the supply and demand constraints (also called rim conditions). (ii) (ii) The number of positive allocations must be equal to $m + n - 1$, where m is the number of rows and n is the number of columns. Any solution that satisfies the above conditions is called non-degenerate basic feasible solution, otherwise, degenerate solution
3.	Complementary slackness property of primal-dual relationship states that for a positive basic variable in the primal, the corresponding dual variable will be equal to zero. Alternatively, for a non-basic variable in the primal (which is zero), the corresponding dual variable will be basic and positive.
4.	In bounded variable simplex method, the optimality condition for a solution is the same as the simplex method, discussed earlier. But the inclusion of constraints $x + s' = u$ in the simplex table requires modification in the feasibility condition of the simplex method due to the following reasons: (i) A basic variable should become a non-basic variable at its upper bound (in usual simplex method all non-basic variables are at zero level). (ii) (ii) When a non-basic variable becomes a basic variable, its value should not exceed its upper bound and should also not disturb the non-negativity and upper bound conditions of all existing basic variables.
5.	Pure (all) integer programming problems is an ILP in which all decision variables are restricted to integer values.
6.	Gomory's algorithm has the following properties. (i) Additional linear constraints never cutoff that portion of the original feasible solution space that contains a feasible integer solution to the original problem. (ii) (ii) Each new additional constraint (or hyperplane) cuts off the current non-integer optimal solution to the linear programming problem.
7.	A necessary condition for a point x_0 to be the local extrema (local maximum and minimum) of a function $y = f(x)$ defined in the interval $a \leq x \leq b$ is that the first derivative of $f(x)$ exists as a finite number at $x = x_0$ and $f'(x_0) = 0$.
8.	$H(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \dots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$
9.	Dynamic programming (DP) differs from linear programming in two ways: (i) In DP, there is no set procedure (algorithm) as in LP to solve any decision-problem. The DP technique allows to break the given problem into a sequence of smaller subproblems, which are then solved in a sequential order (stage). (ii) (ii) LP approach provides one-time period (single stage) solution to a problem whereas DP approach is useful for decision-making over time and solves each subproblem optimally.
10.	The dynamic programming problem can be decomposed or divided into a sequence of smaller subproblems called stages. At each stage there are a number of decision alternatives (courses of action) and a decision is made by selecting the most suitable alternative. Stages represent different time periods in the planning period. For example, in the replacement problem each year is a stage, in the salesman allocation problem each territory represents a stage.

11.	<p>Suppose the <i>primal LP problem</i> is given in the form</p> <p>Maximize $Z_x = c_1x_1 + c_2x_2 + \dots + c_nx_n$ subject to the constraints</p> $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$ $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$ $\vdots \qquad \qquad \qquad \vdots$ $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$ <p>and $x_1, x_2, \dots, x_n \geq 0$</p> <p>Then the corresponding <i>dual LP problem</i> is written as:</p> <p>Minimize $Z_y = b_1y_1 + b_2y_2 + \dots + b_my_m$ subject to the constraints</p> $a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \leq c_1$ $a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \leq c_2$ $\vdots \qquad \qquad \qquad \vdots$ $a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \leq c_n$ <p>and $y_1, y_2, \dots, y_m \geq 0$</p>
12.	$s_g = -f_r + \sum_{j \in R_+} a_{rj} x_j + \left(\frac{f_r}{f_r - 1} \right) \sum_{j \in R_-} a_{rj} x_j$

Part - B

13.	<p>In all mathematical models, assumptions are made for reducing the complex real-world problems into a simplified form that can be more readily analyzed. The following are the major assumptions of an LP model:</p> <ol style="list-style-type: none"> 1. Certainty: In LP models, it is assumed that all its parameters such as: availability of resources, profit (or cost) contribution per unit of decision variable and consumption of resources per unit of decision variable must be known and constant. 2. Additivity: The value of the objective function and the total amount of each resource used (or supplied), must be equal to the sum of the respective individual contribution (profit or cost) of the decision variables. For example, the total profit earned from the sale of two products A and B must be equal to the sum of the profits earned separately from A and B. Similarly, the amount of a resource consumed for producing A and B must be equal to the total sum of resources used for A and B individually. 3. Linearity (or proportionality): The amount of each resource used (or supplied) and its contribution to the profit (or cost) in objective function must be proportional to the value of each decision variable. For example, if production of one unit of a product uses 5 hours of a particular resource, then making 3 units of that product uses $3 \times 5 = 15$ hours of that resource. 4. Divisibility (or continuity): The solution values of decision variables are allowed to assume continuous values. For instance, it is possible to collect 6.254 thousand litres of milk by a milk dairy and such variables are divisible. But, it is not desirable to produce 2.5 machines and such variables are not divisible and therefore must be assigned integer values. Hence, if any of the variable can assume only integer values or are limited to discrete number of values, LP model is no longer applicable.
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14. **Solution** Since the objective function of the primal LP problem is of minimization, change all \leq type constraints to \geq type constraints by multiplying the constraint on both sides by -1 . Also write $=$ type constraint equivalent to two constraints of the type \geq and \leq . Then the given primal LP problem can be rewritten as:

$$\text{Minimize } Z_x = x_1 + 2x_2$$

subject to the constraint

$$(i) -2x_1 - 4x_2 \geq -160,$$

$$(ii) x_1 - x_2 \geq 30$$

$$(iii) x_1 - x_2 \leq 30 \text{ or } -x_1 + x_2 \geq -30, \quad (iv) x_1 \geq 10$$

and

$$x_1, x_2 \geq 0$$

Let y_1, y_2, y_3 and y_4 be the dual variables corresponding to the four constraints in the given order. The dual of the given primal LP problem can then be formulated as follows:

$$\text{Maximize } Z_y = -160y_1 + 30y_2 - 30y_3 + 10y_4$$

subject to the constraints

$$(i) -2y_1 + y_2 - y_3 + y_4 \leq 1,$$

$$(ii) -4y_1 - y_2 + y_3 \leq 2$$

and

$$y_1, y_2, y_3, y_4 \geq 0$$

Let $y = y_2 - y_3$ ($y_2, y_3 \geq 0$). The above dual problem then reduces to the form

$$\text{Maximize } Z_y = -160y_1 + 30y + 10y_4$$

subject to the constraints

$$(i) -2y_1 + y + y_4 \leq 1,$$

$$(ii) -4y_1 - y \leq 2$$

and

$$y_1, y_4 \geq 0; \quad y \text{ unrestricted in sign}$$

15. **Solution Step 1:** Obtain the optimal solution to the LP problem ignoring the integer value restriction by the simplex method.

			$c_j \rightarrow$			
			1	1	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b (= x_B)$	x_1	x_2	s_1	s_2
1	x_1	1/3	1	0	1/3	-2/3
1	x_2	2	0	1	0	1
$Z = 7/2$		$c_j - z_j$	0	0	-1/3	-1/3

In Table 7.1, since all $c_j - z_j \leq 0$, the optimal solution of LP problem is: $x_1 = 1/3$, $x_2 = 2$ and Max $Z = 7/2$.

Step 2: In the current optimal solution, shown in Table 7.1 all basic variables in the basis (x_B -column) did not assume integer value. Thus solution is not desirable. To obtain an optimal solution satisfying integer value requirement, go to step 3.

Step 3: Since x_1 is the only basic variable whose value is a non-negative fractional value, therefore consider first row (x_1 -row) as *source row* in Table 7.1 to generate Gomory cut as follows:

$$\frac{1}{3} = x_1 + 0x_2 + \frac{1}{3}s_1 - \frac{2}{3}s_2 \quad (x_1\text{-source row})$$

The factoring of numbers (integer plus fractional) in the x_1 -source row gives

$$\left(0 + \frac{1}{3}\right) = (1 + 0)x_1 + \left(0 + \frac{1}{3}\right)s_1 + \left(-1 + \frac{1}{3}\right)s_2$$

Each of the non-integer coefficients is factored into integer and fractional parts in such a manner that the fractional part is strictly positive.

Rearranging all of the integer coefficients on the left-hand side, we get

$$\frac{1}{3} + (s_2 - x_1) = \frac{1}{3}s_1 + \frac{1}{3}s_2$$

Since value of variables x_1 and s_2 is assumed to be non-negative integer, left-hand side must satisfy

$$\frac{1}{3} \leq \frac{1}{3}s_1 + \frac{1}{3}s_2 \quad (\text{Ref. Eq. 4})$$

$$\frac{1}{3} + s_{g_1} = \frac{1}{3}s_1 + \frac{1}{3}s_2 \quad \text{or} \quad s_{g_1} - \frac{1}{3}s_1 - \frac{1}{3}s_2 = -\frac{1}{3} \quad (\text{Cut I})$$

where s_{g_1} is the new non-negative (integer) slack variable.

Adding this equation (also called Gomory cut) at the bottom of Table 7.1, the new values so obtained is shown in Table 7.2.

			$c_j \rightarrow$				
			1	1	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}
1	x_1	1/3	1	0	1/3	-2/3	0
1	x_2	2	0	1	0	10	0
0	s_{g_1}	-1/3	0	0	-1/3	-1/3	1 \rightarrow
$Z = 7/2$		$c_j - z_j$	0	0	-1/3	-1/3	0
		Ratio: $\text{Min } (c_j - z_j)/y_{3j} (< 0)$	-	-	1	1	-
					↑		

Step 4: Since the solution shown in Table 7.2 is infeasible, apply the dual simplex method to find a feasible as well as an optimal solution. The key row and key column are marked in Table 7.2. The new solution is obtained by applying the following row operations.

$$R_3(\text{new}) \rightarrow R_3(\text{old}) \times -3; \quad R_1(\text{new}) \rightarrow R_1(\text{old}) - (1/3)R_3(\text{new})$$

The new solution is shown in Table 7.3.

$c_j \rightarrow$			1	1	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}
1	x_1	0	1	0	0	-1	1
1	x_2	2	0	1	0	1	0
0	s_{g_1}	1	0	0	1	1	-3
$Z = 2$		$c_j - z_j$	0	0	0	0	-1

Since all $c_j - z_j \leq 0$ and value of basic variables shown in x_B -column of Table 7.3 is integer, the solution: $x_1 = 0, x_2 = 2, s_{g_1} = 1$ and $\text{Max } Z = 2$, is an optimal basic feasible solution of the given ILP problem.

16. **Solution** The necessary condition for local optimum (maximum or minimum) value is that gradient

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right] = 0$$

That is, $\partial f / \partial x_1 = 1 + x_2 - 2x_1 = 0$, and $\partial f / \partial x_2 = 2 + x_1 - 2x_2 = 0$. The solution of these simultaneous equations, is: $x_0 = (4/3, 5/3)$.

The sufficient condition can be verified by considering the Hessian matrix as follows:

$$\mathbf{H}(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\det A_1 = \left| \frac{\partial^2 f}{\partial x_1^2} \right| = -2, \text{ and } \det A_2 = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} = 4 - 1 = 3$$

Since the signs of the principal minor determinants of $\mathbf{H}(x)$ are alternating, matrix $\mathbf{H}(x)$ is negative definite and the point $x_0 = (4/3, 5/3)$ is the local maximum of the function $f(x)$.

17. The Wolf's method for solving a quadratic programming problem can be summarized in the following steps:
Step 1: Introduce artificial variables A_j ($j = 1, 2, \dots, n$) in the Kuhn-Tucker condition (i). Then we have

$$c_j - \sum_{k=1}^n x_k d_{jk} - \sum_{i=1}^m \lambda_i a_{ij} + \mu_j + A_j = 0$$

For a starting basic feasible solution we shall have $x_j = 0$, $\mu_j = 0$, $A_j = -c_j$ and $s_i^2 = b_i$. However, this solution would be desirable if and only if $A_j = 0$ for all j .

Step 2: Apply Phase I of the simplex method to check the feasibility of the constraints $\mathbf{Ax} \leq \mathbf{b}$. If there is no feasible solution, then terminate the solution procedure, otherwise get an initial basic feasible solution for Phase II. To obtain the desired feasible solution solve the following problem:

$$\text{Minimize } Z = \sum_{j=1}^n A_j$$

subject to the constraints

$$\sum_{k=1}^n x_k d_{jk} + \sum_{i=1}^m \lambda_i a_{ij} - \mu_j + A_j = -c_j, \quad j = 1, 2, \dots, n$$

$$\sum_{j=1}^n a_{ij} x_j + s_i^2 = b_i, \quad i = 1, 2, \dots, m$$

and $\lambda_i, x_j, \mu_j, s_i, A_j \geq 0$ for all i and j

$$\left. \begin{array}{l} \lambda_i s_i = 0 \\ \mu_j x_j = 0 \end{array} \right\} \text{Complementary slackness conditions}$$

Thus, while deciding for a variable to enter into the basis at each iteration, the complementary slackness conditions must be satisfied.

This problem has $2(m+n)$ variables and $(m+n)$ linear constraints, together with $(m+n)$ complementary slackness conditions.

Step 3: Apply Phase II of the simplex method to get an optimal solution to the problem given in Step 2. The solution, so obtained, will also be an optimal solution of the quadratic programming problem.

18. **The General Procedure**

The procedure for solving a problem by using the dynamic programming approach can be summarized in the following steps:

Step 1: Identify the problem decision variables and specify the objective function to be optimized under certain limitations, if any.

Step 2: Decompose (or divide) the given problem into a number of smaller sub-problems (or stages). Identify the state variables at each stage and write down the transformation function as a function of the state variable and decision variable at the next stage.

Step 3: Write down a general recursive relationship for computing the optimal policy. Decide whether to follow the forward or the backward method for solving the problem.

Step 4: Construct appropriate tables to show the required values of the return function at each stage as shown in Table 22.1.

Step 5: Determine the overall optimal policy or decisions and its value at each stage. There may be more than one such optimal policy.

States, s_n ↓	Decision, $d_n \rightarrow$	$f_n(s_n, d_n)$	Optimal Return	Optimal Decision
		d_n	$f_n^*(s_n)$	d_n^*

19.

Solution This LP problem can be considered as a two-stage, two-state problem because there are two decision variables and two constraints. Starting with the second-stage backward, the procedure is as follows:
The optimal value of $f_2(b_1, b_2)$ at the second stage is given by:

$$f_2(b_1, b_2) = \text{Max}_{0 \leq x_2 \leq b} \{7x_2\}$$

where $b_1 = 8$, and $b_2 = 15$. The feasible value of x_2 is a non-negative value that satisfies all the given constraints $x_2 \leq b_1 (= 8)$ and $2x_2 \leq b_2 (= 15)$. Thus, the maximum value of b that x_2 can assume is: $b = \min(8, 15/2) = 15/2$. Therefore,

$$f_2(b_1, b_2) = \text{Max}_{0 \leq x_2 \leq b} \{7x_2\} = 7 \text{Min} \{8 - 2x_1, (15/2) - (5/2)x_1\}$$

and

$$x_2^* = \text{Min} \{8 - x_1, 7.5 - 2.5x_1\}$$

Proceeding backwards to stage 1 ($j = 1$), the recursive relation for optimization can be expressed as:

$$\begin{aligned} f_1(b_1, b_2) &= \text{Max}_{0 \leq x_1 \leq b} \{8x_1 + f_2^*(b_1 - 2x_1, b_2 - (5/2)x_1)\} \\ &= \text{Max}_{0 \leq x_1 \leq 3} \{8x_1 + 7 \text{Min}(8 - 2x_1; (15/2) - (5/2)x_1)\} \end{aligned}$$

where maximization of variable x_1 satisfying the conditions: $2x_1 \leq b (= 8)$ and $5x_1 \leq b_2 (= 15)$ is the minimum of $b = \min(8/2, 15/5) = 3$. Since, the minimum (i.e. zero) of $(8 - 2x_1, 15/2 - 5x_1/2)$ is obtained at $x_1 = 3$ for $0 \leq x_1 \leq 3$, we get:

$$\begin{aligned} f_1^*(b_1, b_2) &= \text{Max} [8x_1 + 7 \text{Min} \{8 - 2x_1; (15/2) - (5x_1/2)\}] \\ &= \text{Max} \{8 \times 3 + 7 \times 0\} = 24, \text{ at } x_1 = 3 \end{aligned}$$

and

$$x_2^* = \text{Min} \{8 - 2x_1, 7.5 - 2.5x_1\} = \text{Min} \{8 - 6, 7.5 - 2.5(3)\} = 0$$

Hence, the optimum solution to the given LP problem is: $x_1 = 3, x_2 = 0$ and $\text{Max } Z = 94$.

PART - C

20.

Solution Adding slack variable, s_1 ; surplus variable, s_2 and artificial variables, A_1 and A_2 in the constraints of the given LP problem, the standard form of the LP problem becomes.

$$\text{Minimize } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$$

subject to the constraints

$$(i) 2x_1 + 4x_2 + s_1 = 12, \quad (ii) 2x_1 + 2x_2 + A_1 = 10 \quad (iii) 5x_1 + 2x_2 - s_2 + A_2 = 10$$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

An initial basic feasible solution: $s_1 = 12, A_1 = 10, A_2 = 10$ and $\text{Min } Z = 10M + 10M = 20M$ is obtained by putting $x_1 = x_2 = s_2 = 0$. It may be noted that the columns that correspond to the current basic variables and form the basis (identity matrix) are s_1 (slack variable), A_1 and A_2 (both artificial variables). The initial basic feasible solution is given in Table 4.26.

Since the value $c_1 - z_1 = 5 - 7M$ is the smallest value, therefore variable x_1 is chosen to enter into the basis (solution mix). To decide a current basic variable to leave the basis, calculate minimum ratio as shown in Table 4.26.

		$c_j \rightarrow$	5	3	0	0	M	M	
Basic Variables	Basic Variables	Basic Variables	x_1	x_2	s_1	s_2	A_1	A_2	Min Ratio
Coefficient	Variables	Value							
c_B	B	$b (= x_B)$							
0	s_1	12	2	4	1	0	0	0	$12/2 = 6$
M	A_1	10	2	2	0	0	1	0	$10/2 = 5$
M	A_2	10	⑤	2	0	-1	0	1	$10/5 = 2 \rightarrow$
$Z = 20M$		z_j	7M	4M	0	-M	M	M	
		$c_j - z_j$	$5 - 7M$	$3 - 4M$	0	M	0	0	
			↑						

Iteration 1: Introduce variable x_1 into the basis and remove A_2 from the basis by applying the following row operations.

$$R_3 (\text{new}) \rightarrow R_3 (\text{old}) + 5 (\text{key element}); \quad R_2 (\text{new}) \rightarrow R_2 (\text{old}) - 2R_3 (\text{new}).$$

$$R_1 (\text{new}) \rightarrow R_1 (\text{old}) - 2R_3 (\text{new}).$$

The improved basic feasible solution is shown in Table 4.27.

			$c_j \rightarrow$					
			5	3	0	0	M	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(=x_B)$	x_1	x_2	s_1	s_2	A_1	Min Ratio x_B/x_2
0	s_1	8	0	16/5	1	2/5	0	$8/(16/5) = 5/2 \rightarrow$
M	A_1	6	0	6/5	0	2/5	1	$6/(6/5) = 5$
5	x_1	2	1	2/5	0	-1/5	0	$2/(2/5) = 5$
$Z = 10 + 6M$		z_j	5	$(6M/5) + 2$	0	$(2M/5) - 1$	M	
		$c_j - z_j$	0	$(-6M/5) + 1$	0	$(-2M/5) + 1$	0	
				↑				

Iteration 2: Since the value of $c_2 - z_2$ in Table 4.27 is the largest negative value, variable x_2 is chosen to replace basic variable s_1 in the basis. Thus, to get an improved basic feasible solution, apply the following row operations:

$$R_1(\text{new}) \rightarrow R_1(\text{old}) \times 5/16 \text{ (key element); } R_2(\text{new}) \rightarrow R_2(\text{old}) - (6/5)R_1(\text{new}).$$

$$R_3(\text{new}) \rightarrow R_3(\text{old}) - (2/5)R_1(\text{new}).$$

The new solution is shown in Table 4.28.

			$c_j \rightarrow$					
			5	3	0	0	M	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(=x_B)$	x_1	x_2	s_1	s_2	A_1	Min Ratio x_B/s_2
3	x_2	5/2	0	1	5/16	1/8	0	$(5/2)/(1/8) = 40$
M	A_1	3	0	0	-3/8	1/4	1	$3/(1/4) = 12 \rightarrow$
5	x_1	1	1	0	-1/8	-1/4	0	
$Z = 25/2 + 3M$		z_j	5	$3 - 3M/8 + 5/16$		$M/4 - 7/8$	M	
		$c_j - z_j$	0	$03M/8 - 5/16$		$-M/4 + 7/8$	0	
						↑		

Iteration 3: Since $c_4 - z_4 < 0$ (negative) in s_2 -column, the current solution is not optimal. Thus, non-basic variable s_2 is chosen to replace artificial variable A_1 in the basis. To get an improved basic feasible solution, apply the following row operations:

$$R_2(\text{new}) \rightarrow R_2(\text{old}) \times 4 \text{ (key element); } R_1(\text{new}) \rightarrow R_1(\text{old}) - (1/8)R_2(\text{new})$$

$$R_3(\text{new}) \rightarrow R_3(\text{old}) + (1/4)R_2(\text{new}).$$

The improved basic feasible solution is shown in Table 4.29.

			$c_j \rightarrow$			
			5	3	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(=x_B)$	x_1	x_2	s_1	s_2
3	x_2	1	0	1	1/2	0
0	s_2	12	0	0	-3/2	1
5	x_1	4	1	0	-1/2	0
$Z = 23$		z_j	5	3	-1	0
		$c_j - z_j$	0	0	1	0

In Table 4.24, all $c_j - z_j \geq 0$. Thus an optimal solution is arrived at with the value of variables as: $x_1 = 4$, $x_2 = 1$, $s_1 = 0$, $s_2 = 12$ and Min $Z = 23$.

21. The revised simplex method can be summarized in the following steps:

Step 1: Express the given problem in standard form Express the given problem in the revised simplex form by considering the objective function as one of the constraints, and adding the slack and surplus variables, if needed, to the inequalities in order to convert them into equalities.

Step 2: Obtain initial basic feasible solution Start with initial basis matrix $B = I_m$ and then find B_1^{-1} and $B_1^{-1} b$ to form the initial revised simplex table as shown in Table 26.1.

Basic Variables	Solution Values $b (= x_B^{(1)})$	Basis Inverse, B_1^{-1}					$y_k^{(1)}$
		$\beta_0^{(1)}$ (= Z)	$\beta_1^{(1)}$ (= s_1)	$\beta_2^{(1)}$ (= s_2)	...	$\beta_m^{(1)}$ (= s_m)	
Z	θ	1	0	0	...	0	$c_k - z_k$
$x_{B_1} = s_1$	b_1	0	1	0	...	0	y_{1k}
$x_{B_2} = s_2$	b_2	0	0	1	...	0	y_{2k}
...
$x_{B_m} = s_m$	b_m	0	0	0	0	1	y_{mk}

Step 3: Select a variable to enter into the basis (key column) For each non-basic variable, calculate $c_j - z_j$ by using the formula:

$$c_j - z_j = c_j - c_B B_1^{-1} a_j^{(1)}$$

where $B_1^{-1} a_j^{(1)}$ represents the product of the first row of B_1^{-1} and successive columns of A not in B_1^{-1} .

- (i) If all $c_j - z_j \leq 0$, then the current basic solution is optimal. Otherwise go to Step 4.
- (ii) If one or more $c_j - z_j$ are positive, then the variable to enter into the basis may be selected by using the formula:

$$c_j - z_j = \text{Max} \{c_j - z_j; c_j - z_j > 0\}$$

Step 4: Select a variable to leave the basis (key row) Calculate $y_k^{(1)} = B_1^{-1} a_k^{(1)} = a_k^{(1)}$; ($k = 1$) where $a_k^{(1)} = [-c_k, a_k]$

If all $y_{ik} \leq 0$, the optimal solution is unbounded. But if at least one $y_{ik} > 0$, then the variable to be removed from the basis is determined by calculating the ratio:

$$\frac{x_{Br}}{y_{rk}} = \text{Min}_i \left\{ \frac{x_{Bi}}{y_{ik}}; y_{ik} > 0 \right\}$$

That is, the vector $\beta_r^{(1)}$ is selected to leave the basis and go to Step 5.

If the minimum ratio is not unique, i.e. the ratio is same for more than one row, then the resulting basic feasible solution will be degenerate. To avoid cycling from taking place, the usual method of resolving the degeneracy is applied.

Step 5: Update the current solution Update the initial table by introducing a non-basic variable $x_k (= a_k^{(1)})$ into the basis and removing basic variable $x_r (= \beta_r^{(1)})$ from the basis.

Repeat Steps 3 to 5 until an optimal solution is obtained or there is an indication for an unbounded solution.

22. **Solution** Relaxing the integer conditions, the optimal non-integer solution to the given integer LP problem obtained by graphical method as shown in Fig. 7.4 is: $x_1 = 1.92$, $x_2 = 2.69$ and $\max Z_1 = 11.91$.

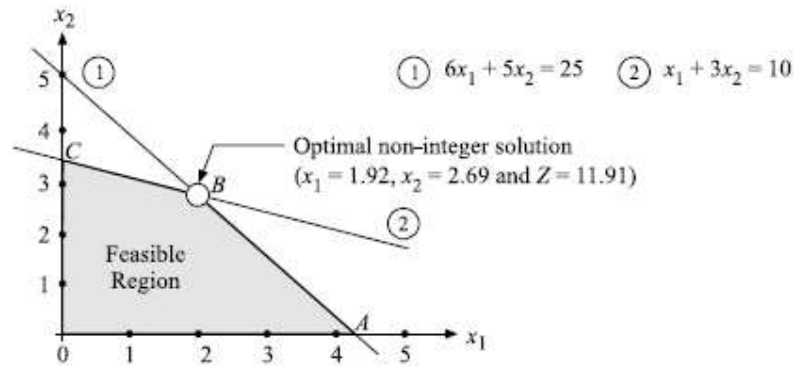
The value of Z_1 represents *initial upper bound* as: $Z_L = 11.91$. Since value of variable x_2 is non-integer, therefore selecting it to decompose (branching) the given problem into two sub-problems by adding two new constraints $x_2 \leq 2$ and $x_2 \geq 3$ to the constraints of original LP problem as follows:

LP Sub-problem B

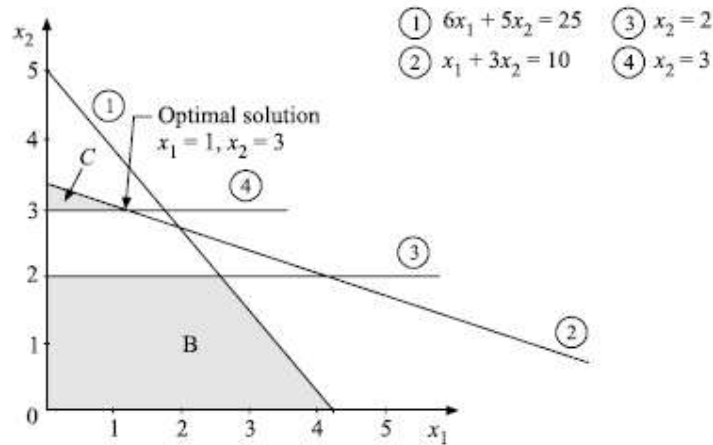
Max $Z = 2x_1 + 3x_2$
 subject to (i) $6x_1 + 5x_2 \leq 25$, (ii) $x_1 + 3x_2 \leq 10$,
 (iii) $x_2 \leq 2$,
 and $x_1, x_2 \geq 0$ integers.

LP Sub-problem C

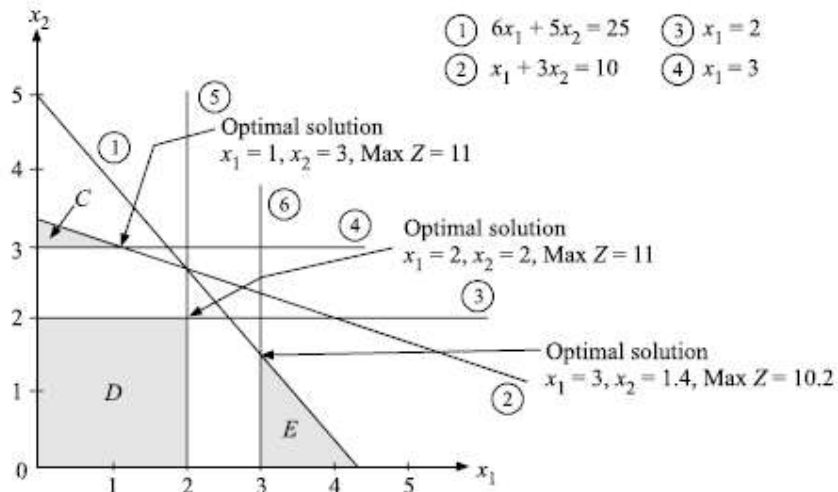
Max $Z = 2x_1 + 3x_2$
 subject to (i) $6x_1 + 5x_2 \leq 25$, (ii) $x_1 + 3x_2 \leq 10$
 (iii) $x_2 \geq 3$,
 and $x_1, x_2 \geq 0$ and integers.



Sub-problem B and C are solved graphically as shown in Fig. 7.5. The feasible solutions are:
 Sub-problem B : $x_1 = 2.5, x_2 = 2$ and $\text{Max } Z_2 = 11$
 Sub-problem C : $x_1 = 1, x_2 = 3$ and $\text{Max } Z_3 = 11$



The value of decision variables at one of the extreme point of feasible region (solution space) of LP sub-problem C, are: $x_1 = 1$ and $x_2 = 3$. Since these are integer values, so there is no need to further decompose (branching) this sub-problem. The value of objective function, $\text{Max } Z_L = 11$ becomes lower bound on the maximum value of objective function, Z for future solutions.



LP sub-problem B is further subdivided into two LP sub-problems D and E (shown in Fig. 7.6) by taking variable $x_1 = 2.5$. Adding two new constraints $x_1 \leq 2$ and $x_1 \geq 3$ to sub-problem B. Also $\text{Max } Z = 11$ is also not inferior to the $Z_L = 11$.

LP Sub-problem D

Max $Z = 2x_1 + 3x_2$
 subject to (i) $6x_1 + 5x_2 \leq 25$, (ii) $x_1 + 3x_2 \leq 10$
 (iii) $x_2 \leq 2$, (iv) $x_1 \leq 2$
 and $x_1, x_2 \geq 0$ and integers.

LP Sub-problem E

Max $Z = 2x_1 + 3x_2$
 subject to (i) $6x_1 + 5x_2 \leq 25$, (ii) $x_1 + 3x_2 \leq 10$,
 (iii) $x_2 \leq 2$, (iv) $x_1 \geq 3$
 and $x_1, x_2 \geq 0$ and integers.

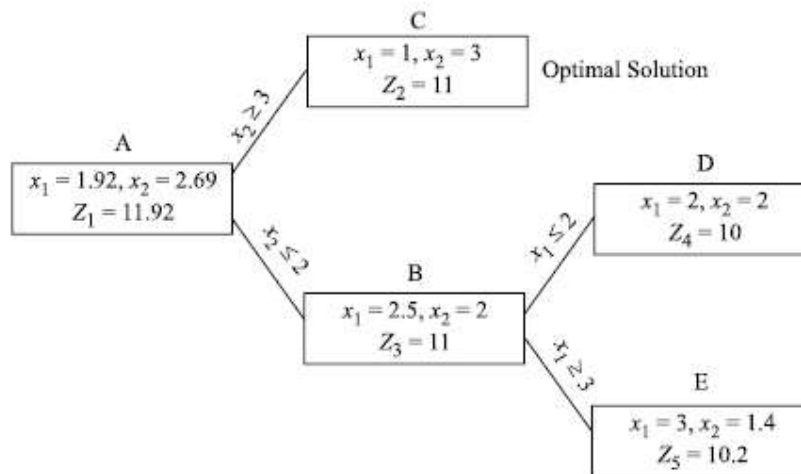
Sub-problems *D* and *E* are solved graphically as shown in Fig. 7.6. The feasible solutions are:

Sub-problem *D* : $x_1 = 2, x_2 = 2$ and max $Z_4 = 10$

Sub-problem *E* : $x_1 = 3, x_2 = 1.4$ and max $Z_5 = 10.2$

The solution of LP sub-problem *D* is satisfying integer value requirement of variables but is inferior to the solution of LP sub-problem *E* in terms of value of objective function, $Z_5 = 10.2$. Hence the value of lower bound $Z_L = 11$ remains unchanged and sub-problem *D* is not considered for further decomposition.

Since the solution of sub-problem *E* is non-integer, it can be further decomposed into two sub-problems by considering variable, x_2 . But the value of objective function ($Z_5 = 10.2$) is inferior to the lower bound and hence this does not give a solution better than the one already obtained. The sub-problem *E* is also not considered for further branching. Hence, the best available solution corresponding to sub-problem *C* is the integer optimal solution: $x_1 = 1, x_2 = 3$ and Max $Z = 11$ of the given integer LP problem. The entire branch and bound procedure for the given Integer LP problem is shown in Fig. 7.7.



23.

Solution Here $f(x_1, x_2) = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

$$g_1(x_1, x_2) = x_2 - 8 \leq 0$$

$$g_2(x_1, x_2) = x_1 + x_2 - 10 \leq 0$$

The Lagrangian function can be formulated as:

$$L(x, s, \lambda) = f(x) - \lambda_1 [g_1(x) + s_1^2] - \lambda_2 [g_2(x) + s_2^2]$$

The Kuhn-Tucker necessary conditions can be stated as:

$$(i) \quad \frac{\partial f}{\partial x_j} - \sum_{i=1}^2 \lambda_i \frac{\partial g_i}{\partial x_j}, \quad j = 1, 2 \qquad (ii) \quad \lambda_i g_i(x) = 0, \quad i = 1, 2$$

$$12 + 2x_2 - 4x_1 - \lambda_2 = 0$$

$$\lambda_1 (x_2 - 8) = 0$$

$$21 + 2x_1 - 4x_2 - \lambda_1 - \lambda_2 = 0$$

$$\lambda_2 (x_1 + x_2 - 10) = 0$$

$$(iii) \quad g_i(x) \leq 0$$

$$(iv) \quad \lambda_i \geq 0, \quad i = 1, 2$$

$$x_2 - 8 \leq 0$$

$$x_1 + x_2 - 10 \leq 0$$

There may arise four cases:

Case 1: If $\lambda_1 = 0, \lambda_2 = 0$, then from Condition (i), we have:

$$12 + 2x_2 - 4x_1 = 0 \quad \text{and} \quad 21 + 2x_1 - 4x_2 = 0$$

Solving these equations, we get $x_1 = 15/2, x_2 = 9$. However, this solution violates condition (iii) and therefore it should be discarded.

Case 2: $\lambda_1 \neq 0, \lambda_2 \neq 0$, then from condition (ii) we have:

$$x_2 - 8 = 0 \quad \text{or} \quad x_2 = 8$$

$$x_1 + x_2 - 10 = 0 \quad \text{or} \quad x_1 = 2$$

Substituting these values in condition (i), we get $\lambda_1 = -27$ and $\lambda_2 = 20$. However, this solution violates the condition (iv) and therefore may be discarded.

Case 3: $\lambda_1 \neq 0, \lambda_2 = 0$, then from conditions (ii) and (i) we have:

$$x_1 + x_2 = 10$$

$$2x_2 - 4x_1 = -12$$

$$2x_1 - 4x_2 = -12 + \lambda_1$$

Solving these equations, we get $x_1 = 2, x_2 = 8$ and $\lambda_1 = -16$. However, this solution violates the condition (iv) and therefore may be discarded.

Case 4: $\lambda_1 = 0, \lambda_2 \neq 0$, then from conditions (i) and (ii) we have:

$$2x_2 - 4x_1 = -12 + \lambda_2$$

$$2x_1 - 4x_2 = -21 + \lambda_2$$

$$x_1 + x_2 = 10$$

Solving these equations, we get $x_1 = 17/4, x_2 = 23/4$ and $\lambda_2 = 13/4$. This solution does not violate any of the Kuhn-Tucker conditions and therefore must be accepted.

Hence, the optimum solution of the given problem is: $x_1 = 17/4, x_2 = 23/4, \lambda_1 = 0$ and $\lambda_2 = 13/4$ and Max $Z = 1734/16$.

24. **Solution** Let us define state variable x_j ($j = 1, 2, 3$) such that:

$$x_3 = u_1 + u_2 + u_3 = 10, \text{ at stage 3}$$

$$x_2 = x_3 - u_3 = u_1 + u_2, \text{ at stage 2}$$

$$x_1 = x_2 - u_2 = u_1, \text{ at stage 1}$$

The maximum value of Z for any feasible value of state variable is given by:

$$f_3(x_3) = \text{Max}_{u_3} \{u_3 \cdot f_2(x_2)\}$$

$$f_2(x_2) = \text{Max}_{u_2} \{u_2 \cdot f_1(x_1)\}$$

$$f_1(x_1) = u_1 = x_2 - u_2$$

Thus,
$$f_2(x_2) = \text{Max}_{u_2} \{u_2 \cdot (x_2 - u_2)\} = \text{Max}_{u_2} \{u_2 x_2 - u_2^2\}$$

Differentiating $f_2(x_2)$ with respect to u_2 and equating to zero (necessary condition for maximum or minimum value of a function), we have: $x_2 - 2u_2 = 0$ or $u_2 = x_2/2$

Now using Bellman's principle of optimality, we get: $f_2(x_2) = (x_2/2) \cdot x_2 - (x_2/2)^2 = x_2^2/4$

and
$$f_3(x_3) = \text{Max}_{u_3} \{u_3 \cdot f_2(x_2)\} = \text{Max}_{u_3} \{u_3 \cdot (x_2^2/4)\} = \text{Max}_{u_3} \left\{ u_3 \cdot \frac{(x_3 - u_3)^2}{4} \right\}$$

Again differentiating $f_3(x_3)$ with respect to u_3 and equating to zero we get:

$$\frac{1}{4} \{u_3 \cdot 2(x_3 - u_3)(-1) + (x_3 - u_3)^2\} = 0$$

$$(x_3 - u_3)(-2u_3 + x_3 - u_3) = 0$$

$$(x_3 - u_3)(x_3 - 3u_3) = 0$$

Now either $u_3 = x_3$, which is trivial as $u_1 + u_2 + u_3 = x_3$ or $x_3 - 3u_3 = 0$, or $u_3 = x_3/3 = 10/3$. Therefore,

$$u_2 = \frac{x_2}{2} = \frac{x_3 - u_3}{2} = \frac{1}{2} \left(10 - \frac{10}{3} \right) = \frac{10}{3}$$

$$u_1 = x_2 - u_2 = \frac{20}{3} - \frac{10}{3} = \frac{10}{3}$$

Thus $u_1 = u_2 = u_3 = 10/3$ and hence $\text{Max} \{u_1 \cdot u_2 \cdot u_3\} = (10/3)^3 = 1,000/27$.

25. **Solution** The given maximization problem can be converted into a minimization problem by subtracting from the largest element (i.e. 41) all the elements of the given table. The new cost data so obtained is given in Table 10.15.

(a)

	A	B	C	D	E
1	9	3	1	13	1
2	1	17	13	20	5
3	0	14	8	11	4
4	19	3	0	5	5
5	12	8	1	6	2

(b)

	A	B	C	D	E
1	8	0	0	7	0
2	0	14	12	14	4
3	0	12	8	6	4
4	19	1	0	0	5
5	11	5	0	0	1

Apply Step 2 of the Hungarian method to get the opportunity cost table as shown in Table 10.15(b).

Make assignments in Table 10.15(b) by applying Hungarian method as shown in Table 10.16(a).

(a)

	A	B	C	D	E	
1	8	0	∞	7	∞	
2	0	14	12	14	4	✓
3	∞	12	8	6	4	✓
4	19	1	∞	0	5	
5	11	5	0	∞	1	
						✓

(b)

	A	B	C	D	E
1	12	0	0	7	0
2	0	10	8	10	0
3	0	8	4	2	0
4	23	1	0	0	5
5	15	5	0	0	1

The solution shown in Table 10.16(a) is not optimal since only four assignments are made. Cover the zeros with the minimum number of lines (= 4) as shown in Table 10.16(a).

Develop the revised cost matrix by selecting the minimum element (= 4) among all uncovered elements by the lines. Subtract 4 from all uncovered elements, including itself, and add it to the element at the intersection of the lines. A revised cost table, so obtained, is shown in Table 10.16(b).

Repeat the above procedure again to make the assignments in the reduced Table 10.16(b). The two alternative assignments are shown in Tables 10.17(a) and (b). Two more alternative solutions exist due to presence of zero element in cells (4, C), (4, D) and cells (5, C), (5, D).

(a)

	A	B	C	D	E
1	12	0	∞	7	∞
2	0	12	10	12	∞
3	∞	10	4	2	0
4	23	1	0	∞	5
5	15	5	∞	0	1

(b)

	A	B	C	D	E
1	12	0	∞	7	∞
2	∞	12	10	12	0
3	0	10	4	2	∞
4	23	1	0	∞	5
5	15	5	∞	0	1

Two alternative optimal assignments are as follows:

Assignment Set I

Salesman	District	Sales (in '000 Rs)
1	B	38
2	A	40
3	E	37
4	C	41
5	D	35
	Total	191

Assignment Set II

Salesman	District	Sales (in '000 Rs)
1	B	38
2	E	36
3	A	41
4	C	41
5	D	35
	Total	191